ALMOST REGULAR FORMS

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By

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CHAPTER I

INTRODUCTION

Associated with the ternary quadratic form

(1) $$f = ax^2 + by^2 + cz^2 + 2ryz + 2sxz + 2txy$$

is the determinant

$$\begin{vmatrix} a & t & s \\ t & b & r \\ s & r & c \end{vmatrix}$$

which is represented by d. The greatest common divisor of the cofactors of the elements of d is denoted by Ω. With f there is associated a reciprocal form $\overset{F}{f} = AX^2 + BY^2 + \cdots + 2TXY$ where ΩA, ΩB, $\cdots$, ΩT are the cofactors of the elements of d.

Two forms of the same determinant are said to be <u>equivalent</u> when one may be transformed into the other by means of a linear transformation of determinant one.

The greatest common divisor, or briefly the g.c.d., of the integers $a_1, a_2, \cdots, a_n$ is denoted by $(a_1, a_2, \cdots, a_n)$. If $(a,b,c,r,s,t) = 1$, f is said to be <u>primitive</u>. When $(a,b,c,r,s,t) = 1$, f is said to be <u>properly</u> or <u>improperly</u> <u>primitive</u> according as (a,b,c) is odd or even.

The integers d and Ω associated with a primitive form f are invariants of f with respect to any linear transformation of determinant one. The integer Δ is defined by

$d = \Omega^2\Delta$. The determinant D associated with F is given by $D = \Omega\Delta^2$. Let Ω'' and Δ'' be the greatest powers of 2 dividing Ω and Δ respectively. Hence Ω' and Δ', defined by $\Omega = \Omega'\Omega''$ and $\Delta = \Delta'\Delta''$, are odd.

A form is positive[1] if and only if

(2) $$a > 0,\ ab - t^2 > 0, \text{ and } d > 0.$$

The form f is said to represent primitively an integer N if f (x,y,z) = N and (x,y,z) = 1.

All of the forms equivalent to a given form are said to form a class of forms. In a class of forms there is one which is unique and this one is called the reduced form of the class.[2]

Two forms having the same invariants Ω and Δ are said to belong to the same genus of forms if their like characters all have the same value.[3]

According to the definitions of class and genus, all forms of the same determinant and invariants Ω and Δ are divided into genera and the genera into classes.

The ternary quadratic form f has associated with it certain arithmetic progressions such that no integer contained in any one of these progressions is represented by the form.[4]

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[1]L. E. Dickson, Studies in the Theory of Numbers, (Chicago), 1930, p. 10

[2]Ibid., p. 179. [3]Ibid., pp. 51,52.

[4]B. W. Jones, A New Definition of Genus for Ternary Quadratic Forms, Transactions of the American Mathematical Society, Vol. 33 (1931), p. 99.

The subject of regular and irregular forms was introduced by L. E. Dickson in 1927.[1] He defined a regular form as being a properly primitive positive ternary quadratic form which represents all integers not included in its arithmetic progressions, and found the possible regular forms for $f = ax^2 + by^2 + cz^2$ where $a = 1$ and $(b,c) = 1$. All properly primitive positive ternary quadratic forms, not regular, were classified by Dickson as being irregular.

B. W. Jones proved there exist exactly 102 regular forms $f = ax^2 + by^2 + cz^2$ with $(a,b,c) = 1$ in his Chicago dissertation of 1928.[2] He and Gordon Pall gave proof of the regularity of these 102 forms by use of quaternions.[3] They also have shown that a genus of more than one class may contain a form which is <u>almost regular</u>. For example $f' = 3x^2 + 4y^2 + 9z^2$ represents no w^2 where odd $w \equiv 1 \pmod 3$. This form represents all numbers outside of its associated arithmetic progressions, hence Jones and Pall referred to this form as being almost regular or semi-regular. They state: "These almost regular forms are new and are one of the most significant products of the method of proof."[4] Jones has shown that a regular form represents

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[1]L. E. Dickson, <u>Ternary Quadratic Forms and Congruences</u>, Annals of Mathematics, Vol. 28 (1927), pp. 333-41.

[2]L. E. Dickson, <u>Modern Elementary Theory of Numbers</u>, (Chicago), 1939, p. 111.

[3]Burton W. Jones and Gordon Pall, <u>Regular and Semi-regular Positive Ternary Quadratic Forms</u>, Acta Mathematica, Vol. 70 (1938), pp. 165-191.

[4]Ibid., p. 167.

all the integers that any form in the genus represents.[1] Hence, obviously, a form is regular if it is in a genus of only one class. There exist 82 genera containing only one class, 19 genera contain 2 classes, and 1 genus contains 4 classes among the 102 regular forms without cross product terms. Of the irregular reduced forms involved in these cases only 9 have been properly classified as being almost regular or semi-regular by use of quaternions.[2]

Definition 1. A properly primitive ternary quadratic form is said to have an almost characteristic of the first kind associated with it if there exist a finite number of integers, not contained in the arithmetic progressions associated with the form, that are not represented primitively by the form.

Definition 2. A properly primitive ternary quadratic form is said to have an almost characteristic of the second kind associated with it, if there exist odd squares, whose positive square roots form an arithmetic progression, and the odd squares are not represented primitively by the form.

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[1]Burton W. Jones, The Regularity of a Genus of Positive Ternary Quadratic Forms, Transactions of the American Mathematical Society, Vol. 33 (1931), p. 124.

[2]Jones and Pall, Regular and Semi-regular Forms, Acta Mathematica, Vol. 70 (1938), p. 191.

Definition 3. A properly primitive ternary quadratic form is said to have _an almost characteristic of the third kind_ associated with it, if there exist odd squares, whose positive square roots form an arithmetic progression, and the odd square multiples of Ω are not represented primitively by the form.

Definition 4. A properly primitive ternary quadratic form which represents all integers not contained in the set of arithmetic progressions associated with it, but possesses an almost characteristic of the first, second, or third kind is said to be almost regular.

Definition 5. An almost regular form is said to be an almost regular form of the first, second, or third class according as it possesses an almost characteristic of the first, second, or third kind respectively.

This dissertation deals with forms possessing an almost characteristic of the second kind and forms possessing an almost characteristic of the third kind. In this dissertation it is shown that there exist infinitely many genera containing two classes of forms possessing almost characteristics of the second and third kinds.

The method of proof follows that used by Burton W. Jones and E. H. Hadlock on indefinite forms.[1]

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[1]Burton W. Jones and E. H. Hadlock, _Properly Primitive Ternary Quadratic Genera of More Than One Class_, Proceedings of the American Mathematical Society, Vol. 4, No. 4 (August, 1953), pp. 539-543.

In their paper it is to be noted that their proof[1] of Lemma 2 is general so that the relation between the integers represented by (1) with $t = 0$ and the integers represented by

(3) $$g = bu^2 + av^2 + dz^2$$

is given by

(4) $$abf = g$$

where the variables u, v, and z are given by

(5) $$\begin{aligned} u &= ax + sz \\ v &= by + rz \\ z &= z. \end{aligned}$$

In addition to the papers listed in the footnotes there have been others by Henry J. S. Smith, Burton W. Jones, and E. H. Hadlock that have been useful in the preparation of this dissertation. These are listed in the bibliography.

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[1] Ibid., pp. 541,542.

CHAPTER II

THE FORM $f = ax^2 + \Omega q^2y^2 + cz^2 + 2\Omega r'yz + 2sxz$ WHERE THE INVARIANTS Ω AND Δ CONTAIN AT LEAST ONE COMMON ODD PRIME FACTOR TO AN ODD POWER

1. Lemma 1.2. Let $\Omega' = \Omega_1^2 P_{\Omega\Delta}$ and $\Delta' = \Delta_1^2 P_{\Omega\Delta}$ where Ω_1^2 and Δ_1^2 are the largest squares dividing Ω' and Δ' respectively. Then $P_{\Omega\Delta}$ is the common factor of Ω' and Δ' which is an odd prime or a product of distinct odd primes. Let p and q be distinct odd primes and prime to the determinant d, and let $P_{\Omega\Delta}$, Ω'', Δ'', p, and q have the values as listed for the particular cases in

TABLE 1

Case	$P_{\Omega\Delta} \equiv$ (mod 4)	Power of 2 Ω''	Power of 2 Δ''	$p \equiv$	$q \equiv$
1	3	even	even	1 (mod 4)	1, 3 (mod 4)
2	3	odd	odd	1 (mod 4)	1, 7 (mod 8)
3	3	odd	even	1, 3(mod 8)	1, 3 (mod 4)
4	3	even	odd	1, 3(mod 8)	1, 7 (mod 8)
5	1	even	even	1 (mod 4)	1 (mod 4)
6	1	odd	odd	1 (mod 4)	1 (mod 8)
7	1	odd	even	1, 3(mod 8)	1 (mod 4)
8	1	even	odd	1, 3(mod 8)	1 (mod 8)

Let $\left(\frac{q}{P_{\Omega\Delta}}\right) = 1$ and let the characters with respect to Ω and the characters with respect to 4 and 8, when they

exist, have the value one. Then a properly primitive positive ternary quadratic form f with a properly primitive reciprocal form F, exists for which $a = 1$ or p^2, $b = \Omega q^2$, $r = \Omega r'$ and $t = 0$. Moreover $\left(\frac{F}{q_\Delta}\right) = 1$ for each odd prime factor q_Δ of Δ.

Take $a = 1$ or p^2. Then the character $\left(\frac{a}{p_\Omega}\right) = 1$ for each odd prime factor p_Ω of Ω. Also $\left(\frac{-1}{a}\right) = 1 = \left(\frac{2}{a}\right)$ since $a = 1$ or p^2.

From the definition of a cofactor of an element of d, $C = aq^2$ so that $\left(\frac{F}{q_\Delta}\right) = \left(\frac{C}{q_\Delta}\right) = 1$. Also $\left(\frac{-1}{C}\right) = \left(\frac{-1}{aq^2}\right) = 1 = \left(\frac{2}{aq^2}\right)$.

From the expression for the determinant d of f there is obtained

(6) $$aA - q^2s^2 = \Omega\Delta$$

where

(7) $$A = q^2c - \Omega r'^2.$$

Integers A and s must be found for which (6) is true, and then integers c and r' are to be found for which (7) is true.

If $a = 1$, then for $s \neq 0$, A is given by (6). In order for (6) to have an integral solution in A and s when $a = p^2$, it is necessary and sufficient for

(8) $$s^2 \equiv -\Omega\Delta q_1 \pmod{p^2},$$

to have a solution where

(9) $$q^2q_1 \equiv 1 \pmod{p^2}.$$

The quadratic character

$$\left(\frac{-\Omega\Delta q_1}{p}\right) = \left(\frac{-\Omega\Delta q^2 q_1}{p}\right) = \left(\frac{-\Omega\Delta}{p}\right) = \left(\frac{-\Omega_1^2 P_{\Omega\Delta}\Omega''\Delta_1^2 P_{\Omega\Delta}\Delta''}{p}\right)$$

$= \left(\frac{-\Omega''\Delta''}{p}\right)$, in cases 1, 2, 5, and 6, becomes $\left(\frac{-1}{p}\right) = 1$, since $p \equiv 1 \pmod 4$, and in cases 3, 4, 7, and 8 becomes $\left(\frac{-2}{p}\right) = 1$, since $p \equiv 1, 3 \pmod 8$.

Hence the congruence (8) has an integral solution for s and the integral value of A is given by (6).

In order that (7) may have an integral solution in c and r', it is necessary and sufficient that

(10) $$r'^2 \equiv -A\Omega_2 \pmod{q^2}$$

has a solution where

(11) $$\Omega\Omega_2 \equiv 1 \pmod{q^2}.$$

From (6) it is seen that $\Omega\Delta \equiv aA \pmod{q^2}$. Hence the value of the quadratic character

$$\left(\frac{-A\Omega_2}{q}\right) = \left(\frac{-A\Omega_2\Omega^2}{q}\right) = \left(\frac{-A\Omega}{q}\right) = \left(\frac{-aA\Omega}{q}\right) = \left(\frac{-\Omega\Delta\Omega}{q}\right) = \left(\frac{-\Delta}{q}\right)$$

$$= \left(\frac{-\Delta_1^2 P_{\Omega\Delta}\Delta''}{q}\right) = \left(\frac{-P_{\Omega\Delta}\Delta''}{q}\right).$$

In cases 1, 3, 5 and 7,

$$\left(\frac{-P_{\Omega\Delta}\Delta''}{q}\right) = \left(\frac{-P_{\Omega\Delta}}{q}\right) = \left(\frac{q}{P_{\Omega\Delta}}\right) = 1.$$

In cases 2, 4, 6 and 8,

$$\left(\frac{-P_{\Omega\Delta}\Delta''}{q}\right) = \left(\frac{-P_{\Omega\Delta}}{q}\right)\left(\frac{2}{q}\right) = \left(\frac{2}{q}\right) = 1,$$

since $q \equiv 1, 7 \pmod 8$. Hence the congruence (10) has an integral solution for r' and from (7) the integral value of c is given by

(12) $$c = \frac{A + \Omega r'^2}{q^2} .$$

Since $(a, b) = 1$, f is properly primitive, and since the relations (2) hold, f is positive.

It is necessary to show that Ω is the greatest common divisor of the cofactors of the elements of d. The expressions for the cofactors of a, b, c, r, s, and t, with $t = 0$, $b = \Omega q^2$, and $r = \Omega r'$ are

(13) $$\begin{aligned} \Omega A &= \Omega q^2 c - \Omega^2 r'^2 \\ \Omega B &= ac - s^2 \\ \Omega C &= a\Omega q^2 \\ \Omega R &= -a\Omega r' \\ \Omega S &= -\Omega q^2 s \\ \Omega T &= \Omega r's \end{aligned}$$

A, C, R, S, and T are obviously integers.

To show that B is an integer, suppose that B is rational and not an integer. Since $ac - s^2$ is an integer the denominator of B divides Ω for $\Omega B = ac - s^2$. From the expression for the determinant of f it follows that

$b\Omega B - ar^2 = \Omega^2\Delta$ and hence $q^2B = ar'^2 + \Delta$. But $ar'^2 + \Delta$ is an integer. Hence the denominator of B divides q^2. But $(q^2, \Omega) = 1$. Hence B is an integer. It follows that Ω is the greatest common divisor of the cofactors of the elements of d when it is shown that F is primitive. A common factor σ of the coefficients of F must divide $C = aq^2$. From the expression for the determinant of F it follows that σ divides $\Omega\Delta^2$. But $(aq^2, \Omega\Delta^2) = 1$. Hence $\sigma = 1$, and F is primitive. Therefore Ω is the greatest common divisor of the cofactors of the elements of d. Further, since $C = aq^2$ is odd, F is properly primitive. Hence Lemma 1.2 is true and there exists a properly primitive positive ternary quadratic form

(14) $$f = ax^2 + \Omega q^2y^2 + cz^2 + 2\Omega r'yz + 2sxz,$$

with a properly primitive reciprocal form F, for each case of Table 1.

2. Lemma 2.2. Let $\Omega' = \Omega_1^2 P_{\Omega\Delta}$ and $\Delta' = \Delta_1^2 P_{\Omega\Delta}$ where Ω_1^2 and Δ_1^2 are the largest squares dividing Ω' and Δ' respectively. Let $P_{\Omega\Delta}$, Ω'', Δ'', p, and q of Lemma 1.2 and odd $\gamma > 0$ have the values as listed for the particular cases in Table 2. Let

$$\left(\frac{\gamma}{P_{\Omega\Delta}}\right) = -\left(\frac{p}{P_{\Omega\Delta}}\right)$$

with $$(\gamma, 2pq\Omega\Delta) = 1.$$

Then f of Lemma 1.2 represents primitively no γ^2.

TABLE 2

Case	$P_{\Omega\Delta} \equiv$ (mod 4)	Power of 2 Ω''	Δ''	$p \equiv 1$	$q \equiv$	$\gamma \equiv$ (mod 8)	$\Omega''\Delta''$
1	3	even	even	(mod 4)	1,3(mod 4)	. . .	$\geq$ 1
2	3	odd	odd	(mod 8)	1,7(mod 8)	$\pm$ pq	$\geq$ 64
3	1	even	even	(mod 4)	1 (mod 4)	pq	$\geq$ 64
4	1	odd	odd	(mod 8)	1 (mod 8)	pq	$\geq$ 64

The relation between the integers represented by (14) and the integers represented by (3) is given by (4), where the variables u, v, and z are given by (5).

If $f = \gamma^2$ has a primitive solution (x,y,z), then u, v, and z can have no common prime factors except p and q. Suppose a prime g divides u, v, and z and is prime to p and q. Then it follows from (5) that g divides x, y, and z and the solution is not primitive.

If $\left(\frac{p}{P_{\Omega\Delta}}\right) = -1$, p does not divide z, since $f = \gamma^2$, $z \equiv 0 \pmod p$ implies

(15) $$\gamma^2 \equiv \Omega q^2 y^2 \pmod p.$$

But $\left(\frac{\Omega q^2 y^2}{p}\right) = \left(\frac{\Omega}{p}\right) = \left(\frac{\Omega_1^2 P_{\Omega\Delta}\Omega''}{p}\right) = \left(\frac{P_{\Omega\Delta}\Omega''}{p}\right) = \left(\frac{p}{P_{\Omega\Delta}}\right)\left(\frac{\Omega''}{p}\right)$ $= -\left(\frac{\Omega''}{p}\right)$. In cases 1 and 3, $-\left(\frac{\Omega''}{p}\right) = -\left(\frac{1}{p}\right) = -1$. In cases 2 and 4, $-\left(\frac{\Omega''}{p}\right) = -\left(\frac{2}{p}\right) = -1$. Hence the congruence (15) has no primitive solution and p does not divide z.

From (4) and (5) with $a = p^2$ and $f = \gamma^2$, there is obtained

(16) $$p^2q^2\gamma^2 = q^2u^2 + p^2\Omega v_1^2 + \Omega\Delta z^2, \; v = \Omega v_1,$$

so that

(17) $$(qu)^2 + \Omega(pv_1)^2 = (pq\gamma)^2 - \Omega\Delta z^2,$$

and

(18) $$(qu)^2 + \Omega(pv_1)^2 = \rho\sigma,$$

where

(19) $$\rho = pq\gamma + \sqrt{\Omega\Delta}z$$
$$\sigma = pq\gamma - \sqrt{\Omega\Delta}z$$

Take $\theta = \rho$ or σ. Then ρ and σ must have the same sign since the left member of (18) is positive. From (19) $\rho + \sigma = 2pq\gamma$. Hence ρ and σ are each positive and $\theta > 0$.

Assume there exist integers u, v_1, and z that satisfy (18). Then u, v_1, and θ must satisfy

(20) $$(qu)^2 \equiv -\Omega(pv_1)^2 \pmod{\theta}.$$

The character

$$\left(\frac{-\Omega(pv_1)^2}{\theta}\right) = \left(\frac{-\Omega}{\theta}\right) = \left(\frac{-\Omega_1^2 P_{\Omega_\Delta}\Omega''}{\theta}\right) = \left(\frac{-P_{\Omega_\Delta}\Omega''}{\theta}\right).$$

For case 1,

$$\left(\frac{-P_{\Omega_\Delta}\Omega''}{\theta}\right) = \left(\frac{-P_{\Omega_\Delta}}{\theta}\right) = \left(\frac{\theta}{P_{\Omega_\Delta}}\right),$$

since $P_{\Omega_\Delta} \equiv 3 \pmod{4}$. From (19),

$$\left(\frac{\theta}{P_{\Omega_\Delta}}\right) = \left(\frac{pq\gamma}{P_{\Omega_\Delta}}\right) = -1,$$

because $\left(\frac{\gamma}{P_{\Omega\Delta}}\right) = -\left(\frac{p}{P_{\Omega\Delta}}\right)$ and $\left(\frac{q}{P_{\Omega\Delta}}\right) = 1$.

For case 2,

$$\left(\frac{-P_{\Omega\Delta}\Omega''}{\theta}\right) = \left(\frac{-P_{\Omega\Delta}}{\theta}\right)\left(\frac{\Omega''}{\theta}\right) = \left(\frac{\theta}{P_{\Omega\Delta}}\right)\left(\frac{2}{\theta}\right),$$

since $\left(\frac{-P_{\Omega\Delta}}{\theta}\right)$ has the same value as in case 1. From (19), $\theta \equiv pq\gamma \pmod{P_{\Omega\Delta}}$ and $\theta \equiv \pm 1 \pmod 8$. Hence

$$\left(\frac{\theta}{P_{\Omega\Delta}}\right)\left(\frac{2}{\theta}\right) = \left(\frac{pq\gamma}{P_{\Omega\Delta}}\right)\left(\frac{2}{\theta}\right) = -\left(\frac{2}{\theta}\right) = -1.$$

For case 3,

$$\left(\frac{-P_{\Omega\Delta}\Omega''}{\theta}\right) = \left(\frac{-P_{\Omega\Delta}}{\theta}\right).$$

From (19), $\theta \equiv 1 \pmod 8$ and $\theta \equiv pq\gamma \pmod{P_{\Omega\Delta}}$.

Hence $\left(\frac{-P_{\Omega\Delta}}{\theta}\right) = \left(\frac{\theta}{P_{\Omega\Delta}}\right) = \left(\frac{pq\gamma}{P_{\Omega\Delta}}\right) = -1$.

For case 4,

$$\left(\frac{-P_{\Omega\Delta}\Omega''}{\theta}\right) = \left(\frac{-P_{\Omega\Delta}}{\theta}\right)\left(\frac{2}{\theta}\right) = \left(\frac{\theta}{P_{\Omega\Delta}}\right)\left(\frac{2}{\theta}\right) = -1,$$

since from case 3, $\left(\frac{\theta}{P_{\Omega\Delta}}\right) = -1$, and $\theta \equiv 1 \pmod 8$. Therefore (20) has no primitive integral solution if

$$(\Omega(pv_1)^2, \theta) = 1.$$

It has just been shown that $\left(\frac{-\Omega}{\theta}\right) = -1$, where $(\Omega, \theta) = 1$. Hence there exists an odd prime factor p_θ of θ, prime to Ω, and dividing θ to an odd power such that

$$\left(\frac{-\Omega}{p_\theta}\right) = -1.$$

But by assumption,

(21) $$(qu)^2 \equiv -\Omega\,(pv_1)^2 \pmod{p_\theta}$$

has a solution. Hence

(22) $$qu \equiv pv_1 \equiv 0 \pmod{p_\theta}.$$

By (19) it is seen that when $\left(\frac{p}{P_{\Omega\Delta}}\right) = -1$, $p_\theta \neq p$, since it has been shown that $z \not\equiv 0 \pmod p$. Also when $\left(\frac{p}{P_{\Omega\Delta}}\right) = 1$, $p_\theta \neq p$, because $\left(\frac{p_\theta}{P_{\Omega\Delta}}\right) = -1$. Since $\left(\frac{p_\theta}{P_{\Omega\Delta}}\right) = -1$ and $\left(\frac{q}{P_{\Omega\Delta}}\right) = 1$ it follows that $p_\theta \neq q$. Hence (22) implies

(23) $$u \equiv v_1 \equiv 0 \pmod{p_\theta}.$$

If p_θ divides z, then it follows from (5) that p_θ is a common factor of x, y, and z and hence by (14) f does not represent γ^2 primitively. Therefore $(p_\theta, z) = 1$. If p_θ divided both ρ and σ it would divide $pq\gamma$ and $\Omega\Delta z$ which has just been shown to be impossible. Hence p_θ occurs to an odd power in the right member of (18) and to an even power in the left member which is impossible. Therefore the assumption that (20) has a primitive solution is false and hence there exists no primitive solution to $f = \gamma^2$. Therefore Lemma 2.2 is true.

3. Theorem 1.2. Let $\Omega' = \Omega_1^2 P_{\Omega\Delta}$ and $\Delta' = \Delta_1^2 P_{\Omega\Delta}$ where Ω_1^2 and Δ_1^2 are the largest squares dividing Ω' and Δ' respectively. Let $P_{\Omega\Delta}$, Ω'', Δ'', p, q, and odd $\gamma > 0$ have the values as listed for the particular cases in Table 2.

Let p and q be distinct primes and prime to twice the determinant d with $\left(\frac{q}{P_{\Omega\Delta}}\right) = 1$. Let the quadratic characters with respect to Ω and the characters with respect to 4 and 8 have the value one. Then there exist genera of properly primitive positive ternary quadratic forms, with properly primitive reciprocals, containing at least two classes of forms $f = p^2x^2 + \Omega q^2y^2 + cz^2 + 2\Omega r'yz + 2sxz$ having an almost characteristic of the second kind.

For f of Lemma 1.2 take $a = p^2$ and define $f_1 = f$, when $\left(\frac{p}{P_{\Omega\Delta}}\right) = 1$. For f_1 and f_2 take $b = \Omega q^2$, $r = \Omega r'$, $t = 0$ and then s, r', and c are determined as in Lemma 1.2. From Lemma 1.2 $\left(\frac{F}{q_\Delta}\right) = 1$ depends only upon Δ so that f_1 and f_2 belong to the same genus. By Lemma 2.2,

(24) $$f_1 \neq \gamma^2$$

when $\left(\frac{\gamma}{P_{\Omega\Delta}}\right) = -\left(\frac{p}{P_{\Omega\Delta}}\right) = 1$, and

(25) $$f_2 \neq \gamma^2$$

when $\left(\frac{\gamma}{P_{\Omega\Delta}}\right) = -\left(\frac{p}{P_{\Omega\Delta}}\right) = -1$. Therefore the genus containing f_1 and f_2 contains at least two classes having the almost characteristic of the second kind since there is no odd square represented primitively by both f_1 and f_2.

Since Ω and Δ may have different values subject to the restrictions imposed upon them in the hypothesis, it follows genera of properly primitive positive ternary

quadratic forms exist, with properly primitive reciprocals, containing at least two classes of forms

$$f = p^2x^2 + \Omega q^2y^2 + cz^2 + 2\Omega r'yz + 2sxz$$

having an almost characteristic of the second kind.

4. Lemma 3.2. Properly primitive ternary quadratic forms represent primitively the same integers.

Let the linear transformation

$$\begin{aligned} x &= A_1X + B_1Y + C_1Z \\ y &= A_2X + B_2Y + C_2Z \\ z &= A_3X + B_3Y + C_3Z \end{aligned} \tag{26}$$

of determinant one transform f_1 (x, y, z) into f_2 (X, Y, Z). Now (x, y, z) = 1 implies (X, Y, Z) = 1 since by (26) a common divisor of X, Y, and Z would divide x, y, and z contrary to (x, y, z) = 1.

Also the inverse transformation of (26) is of the form

$$\begin{aligned} X &= a_1x + b_1y + c_1z \\ Y &= a_2x + b_2y + c_2z \\ Z &= a_3x + b_3y + c_3z \end{aligned} \tag{27}$$

and similarly (X, Y, Z) = 1 implies (x, y, z) = 1. Hence if f_1 has an almost characteristic of the second kind, then f_2 has the identical almost characteristic of the second kind.

5. Example 1.2. Let $f = p^2x^2 + \Omega q^2y^2 + cz^2 + 2\Omega r'yz + 2sxz$, $\Omega = 3$, and $\Delta = 12$. Then $P_{\Omega_\Delta} = 3$, Ω'' and Δ'' are even powers of 2 and case 1 of Table 2 applies. Take $p = 5$ and $q = 7$. Hence $f_1 = f \neq \gamma^2$, where $\left(\frac{\gamma}{P_{\Omega\Delta}}\right) = -\left(\frac{p}{P_{\Omega\Delta}}\right) = 1$ so that odd $\gamma \equiv 1 \pmod 3$. It is found that $s = 6$, $r' = 5$ and $c = 3$. Hence

$$f_1 = 25x^2 + 147y^2 + 3z^2 + 30yz + 12xz.$$

The linear transformation of determinant one

$$\begin{pmatrix} 0 & 2 & 3 \\ 0 & 1 & 1 \\ -1 & -9 & -11 \end{pmatrix}$$

transforms f_1 into the reduced form

$$f_3 = 3X^2 + 4Y^2 + 9Z^2.$$

Hence by Lemma 3.2 the class of forms represented by f_3 represents primitively no γ^2, where odd $\gamma \equiv 1 \pmod 3$.

Now take $p = 13$ so that $f_2 = f \neq \gamma^2$, where odd $\gamma \equiv 2 \pmod 3$. Then $s = 60$, $r' = 17$, and $c = 39$. Hence

$$f_2 = 169x^2 + 147y^2 + 39z^2 + 102yz + 120xz.$$

The linear transformation of determinant one

$$\begin{pmatrix} -1 & 6 & -24 \\ -1 & 6 & -23 \\ 3 & -17 & 67 \end{pmatrix}$$

transforms f_2 into the reduced form

$$f_4 = X^2 + 3Y^2 + 36Z^2.$$

Hence by Lemma 3.2 the class of forms represented by f_4 represents primitively no γ^2 where odd $\gamma \equiv 2 \pmod 3$. The characters that exist for 4 and 8 are

$$\left(\frac{25}{3}\right) = \left(\frac{169}{3}\right) = 1$$

$$\left(\frac{-1}{25\cdot 49}\right) = \left(\frac{-1}{169\cdot 49}\right) = 1.$$

Hence f_1 and f_2 are in the same genus of forms. Each possess an almost characteristic of the second kind.

Example 2.2. Let $f = p^2x^2 + \Omega q^2y^2 + cz^2 + 2\Omega r'yz + 2sxz$, $\Omega = 12$ and $\Delta = 3$. Then $P_{\Omega\Delta} = 3$, Ω'' and Δ'' are even powers of 2 and case 1 of Table 2 applies. Take $p = 5$ and $q = 7$. Then $f_1 = f \neq \gamma^2$ where γ odd $\equiv 1 \pmod 3$. It is found that $s = 6$, $r' = 22$, and $c = 120$. Hence

$$f_1 = 25x^2 + 588y^2 + 120z^2 + 528yz + 12xz.$$

The linear transformation

$$\begin{pmatrix} 2 & -3 & 0 \\ 4 & -5 & 1 \\ -9 & 11 & -2 \end{pmatrix}$$

reduces f_1 to

$$f_3 = 4X^2 + 9Y^2 + 12Z^2.$$

Hence by Lemma 3.2 the class of forms represented by f_3 represent primitively no γ^2 where odd $\gamma \equiv 1 \pmod 3$. Therefore this class of forms possesses an almost characteristic of the second kind.

Now take $p = 13$. Then $s = 60$, $r' = 16$, and $c = 84$. Hence

$$f_2 = 169x^2 + 588y^2 + 84z^2 + 384yz + 120xz.$$

This form reduces to

$$f_4 = X^2 + 12Y^2 + 36Z^2.$$

Hence the class of forms represented by f_4 represents primitively no γ^2 where odd $\gamma \equiv 2 \pmod 3$. Therefore this class of forms possesses an almost characteristic of the second kind.

The character $\left(\dfrac{a}{p_\Omega}\right) = \left(\dfrac{25}{3}\right) = \left(\dfrac{169}{3}\right) = 1.$

The character $\left(\dfrac{-1}{a}\right) = \left(\dfrac{-1}{25}\right) = \left(\dfrac{-1}{169}\right) = 1.$

No other characters exist with respect to 4 and 8. Hence f_3 and f_4 are in the same genus of forms.

Example 3.2. Take f as in Example 1.2. Let $\Omega = 48$ and $\Delta = 3$. Then $P_{\Omega\Delta} = 3$. Let $\left(\dfrac{\gamma}{P_{\Omega\Delta}}\right) = 1$. Take $p = 5$ and $q = 7$. Then $s = 12$, $r' = 22$, and $c = 480$. Hence

$$f_1 = 25x^2 + 2352y^2 + 480z^2 + 2112yz + 24xz$$

which reduces to

$$f_3 = 16X^2 + 25Y^2 + 25Z^2 + 14YZ + 16XZ + 16XY.$$

Hence the class of forms represented by f_3 represents primitively no γ^2 where odd $\gamma \equiv 1 \pmod 3$ and this class possesses an almost characteristic of the second kind.

Now take $p = 13$, then

$$f_2 = 169x^2 + 2352y^2 + 265z^2 + 1536yz + 98xz.$$

This reduces to

$$f_4 \cdot = 4X^2 + 48Y^2 + 49Z^2 + 48YZ + 4XZ.$$

Therefore the class of forms represented by f_4 represents primitively no γ^2 where odd $\gamma \equiv 2 \pmod 3$. Hence this class possesses an almost characteristic of the second kind.

The character $\left(\frac{a}{p_\Omega}\right) = \left(\frac{25}{3}\right) = \left(\frac{169}{3}\right) = 1.$

The character $\left(\frac{-1}{a}\right) = \left(\frac{-1}{25}\right) = \left(\frac{-1}{169}\right) = 1.$

The character $\left(\frac{2}{a}\right) = \left(\frac{2}{25}\right) = \left(\frac{2}{169}\right) = 1.$

Hence these two classes of forms are in the same genus.

CHAPTER III

THE FORM $f = ax^2 + \Omega q^2y^2 + cz^2 + 2\Omega r'yz + 2sxz$ WHERE THE INVARIANTS Ω AND Δ CONTAIN NO COMMON ODD PRIME FACTOR TO AN ODD POWER

1. Lemma 1.3. Let $\Omega' = \Omega_1^2$ and $\Delta' = \Delta_1^2$. Let the characters with respect to Ω and the characters with respect to 4 and 8, when they exist, have the value one. Let p and q be distinct odd primes and prime to the determinant d, and let Ω'', Δ'', p, and q have the properties listed for the particular cases in

TABLE 3

Case	Power of 2		$p \equiv$	$q \equiv$	Ω''	Δ''	$\Omega''\Delta''$
	Ω''	Δ''					
1	even	even	1 (mod 4)	1 (mod 4)	$=1$	≥ 4	≥ 4
2	even	even	1 (mod 4)	1 (mod 4)	≥ 4	$= 1$	≥ 4
3	even	even	1 (mod 4)	1 (mod 4)	$= 1$	$= 1$	$= 1$
4	even	even	1 (mod 4)	1 (mod 4)	≥ 4	≥ 4	≥ 16
5	odd	odd	1 (mod 4)	1, 3 (mod 8)	≥ 2	≥ 2	≥ 4
6	odd	even	1, 3 (mod 8)	1 (mod 4)	≥ 2	$= 1$	≥ 2
7	odd	even	1, 3 (mod 8)	1 (mod 4)	≥ 2	≥ 4	≥ 8
8	even	odd	1, 3 (mod 8)	1, 3 (mod 8)	$= 1$	≥ 2	≥ 2
9	even	odd	1, 3 (mod 8)	1, 3 (mod 8)	≥ 4	≥ 2	≥ 8

Then properly primitive positive ternary quadratic forms f and F exist for which $a = 1$ or p^2, $b = \Omega q^2$, $r = \Omega r'$, and $t = 0$. Moreover $\left(\frac{F}{q_\Delta}\right) = 1$ for each odd prime factor q_Δ of Δ.

Take $a = 1$ or p^2. Then the character $\left(\frac{a}{p_\Omega}\right) = 1$ for each odd prime factor p_Ω of Ω. Also $\left(\frac{-1}{a}\right) = 1 = \left(\frac{2}{a}\right)$.

From the definition of a cofactor of an element of d, $C = aq^2$ so that $\left(\frac{-1}{C}\right) = \left(\frac{-1}{aq^2}\right) = 1 = \left(\frac{2}{aq^2}\right)$.

Also $\left(\frac{F}{q_\Delta}\right) = \left(\frac{C}{q_\Delta}\right) = 1$.

From (8) and (9)

$$\left(\frac{-\Omega\Delta q_1}{p}\right) = \left(\frac{-\Omega\Delta q^2 q_1}{p}\right) = \left(\frac{-\Omega\Delta}{p}\right) = \left(\frac{-\Omega''\Delta''}{p}\right).$$

For cases 1 - 5,

$$\left(\frac{-\Omega''\Delta''}{p}\right) = \left(\frac{-1}{p}\right) = 1,$$

since $p \equiv 1 \pmod 4$. For cases 6 - 9,

$$\left(\frac{-\Omega''\Delta''}{p}\right) = \left(\frac{-2}{p}\right) = 1$$

since $p \equiv 1, 3 \pmod 8$. Hence (8) has a solution for s and the integral value of A is given by (6). Also A will have the same parity as the choice of s in all cases except in case 3, where the parity of A is oppositethat of the choice of s.

From (6) it is seen that $\Omega\Delta \equiv aA \pmod{q^2}$. Also

(11) holds. Hence from (10) and (11),

$$\left(\frac{-A\Omega_2}{q}\right) = \left(\frac{-A\Omega^2\Omega_2}{q}\right) = \left(\frac{-A\Omega}{q}\right) = \left(\frac{-aA\Omega}{q}\right) = \left(\frac{-\Omega\Delta\Omega}{q}\right)$$

$$= \left(\frac{-\Delta}{q}\right) = \left(\frac{-\Delta'\Delta''}{q}\right) = \left(\frac{-\Delta''}{q}\right).$$

For cases 1, 2, 3, 4, 6, and 7, $\left(\frac{-\Delta''}{q}\right) = \left(\frac{-1}{q}\right) = 1$, since $q \equiv 1 \pmod 4$.

For cases 5, 8, and 9, $\left(\frac{-\Delta''}{q}\right) = \left(\frac{-2}{q}\right) = 1$, since $q \equiv 1, 3 \pmod 8$. Hence (10) has a solution in r', and the value of c is given by (7), where its parity is the same as A in all casea, except in cases 1, 3, and 8, when r' is taken odd.

Since $(a,b) = 1$, f is primitive, and since a is odd, f is properly primitive. Also f is positive since (2) holds.

That Ω is the greatest common divisor of the cofactors of the elements of d, and that F is properly primitive, follow by arguments identical with those given in Lemma 1.2. Hence Lemma 1.3 is true.

2. Lemma 2.3. If $(\gamma pq\Omega\Delta) = 1$; if $\gamma \equiv 1, 3 \pmod 8$, when $q \equiv 5p, 7p \pmod 8$; if $\gamma \equiv 5, 7 \pmod 8$, when $q \equiv p, 3p \pmod 8$; and if $\Omega''\Delta'' \geq 64$, then in case 5 of Lemma 1.3 where $p \equiv 1 \pmod 4$ f represents primitively no γ^2.

If $f = \gamma^2$ has a primitive solution (x, y, z) then the fact that u, v, z can have no common prime factor other

than p and q follows by precisely the same argument as that given in Lemma 2.2. If $p \equiv 5 \pmod 8$, p does not divide z, since $f = \gamma^2$, $z \equiv 0 \pmod p$ implies (15). But

$$\left(\frac{\Omega q^2 y^2}{p}\right) = \left(\frac{2}{p}\right) = -1,$$

and the congruence (15) has no solution.

From (4) and (5) with $a = p^2$ and $f = \gamma^2$ equations (16), (17), (18), and (19) are obtained. Taking $\theta = \rho$ or σ it follows that $\theta > 0$ by an argument identical with that given in Lemma 2.2. By (19), $\theta \equiv pq\gamma \equiv 5, 7 \pmod 8$.

Assume there exists integers u, v_1, and z that satisfy (18), then u, v_1, and θ must satisfy (20). But $\left(\frac{-\Omega(pv_1)^2}{\theta}\right) = \left(\frac{-\Omega}{\theta}\right) = \left(\frac{-\Omega''}{\theta}\right) = \left(\frac{-2}{\theta}\right) = -1$, since $\theta \equiv 5, 7 \pmod 8$. Hence (20) has no solution if $(\Omega(pv_1)^2, \theta) = 1$. Since $(\Omega, \theta) = 1$ and $\left(\frac{-\Omega}{\theta}\right) = -1$, there exists an odd prime factor p_θ prime to Ω and dividing θ to an odd power such that $\left(\frac{-\Omega}{p_\theta}\right) = -1$. Therefore $\left(\frac{-2}{p_\theta}\right) = -1$ and $p_\theta \equiv 5, 7 \pmod 8$. Hence by assumption (21) has a solution with $(\Omega, p_\theta) = 1$ and $(-\Omega(pv_1)^2, p_\theta) > 1$. Therefore (22) holds. But by (19) when $p \equiv 5 \pmod 8$, $p_\theta \neq p$ since it has been shown that $z \not\equiv 0 \pmod p$. When $p \equiv 1 \pmod 8$, $p_\theta \neq p$ since $p_\theta \equiv 5, 7 \pmod 8$. Also $p_\theta \neq q$, since $q \equiv 1, 3 \pmod 8$. Hence (23) holds. It follows that $(p_\theta, z) = 1$ and that (20)

has no solution by the precise arguments given in Lemma 2.2. Therefore Lemma 2.3 is true.

3. Theorem 1.3. Let Ω' and Δ' be odd squares, and Ω'' and Δ'' each be odd powers of 2 with $\Omega''\Delta'' \geq 64$, and $P_{\Omega_\Delta} = 1$. Let p and q be distinct primes and prime to d with $p \equiv 1 \pmod 4$ and $q \equiv 1, 3 \pmod 8$. Let each of the characters with respect to Ω and the characters with respect to 4 and 8, when they exist, have the value one. Then there exist genera of properly primitive positive ternary quadratic forms $f = p^2x^2 + \Omega q^2y^2 = cz^2 + 2\Omega r'yz + 2sxz$, with properly primitive reciprocals, containing at least two classes of forms with an almost characteristic of the second kind.

For f of Lemma 1.3 define $f_1 = f$ when $b = \Omega q^2$, $q \equiv 5p, 7p \pmod 8$ and $f_2 = f$ when $b = \Omega q^2$, $q \equiv p, 3p \pmod 8$. For f_1 and f_2 take $a = p^2$, $p \equiv 1 \pmod 4$, $t = 0$, and then s, r', and c are determined as in Lemma 1.3.

From Lemma 1.3 it is seen that $\left(\frac{F}{q_\Delta}\right) = \left(\frac{c}{q_\Delta}\right) = 1$ depends only upon Δ so that f_1 and f_2 belong to the same genus of forms. By Lemma 2.3 f_1 represents primitively no γ^2 where $\gamma \equiv 1, 3 \pmod 8$, and f_2 represents primitively no γ^2 where $\gamma \equiv 5, 7 \pmod 8$. Therefore the genus containing f_1 and f_2 contains at least two classes for there exists no odd square represented primitively by both f_1 and f_2. Further, f_1 and f_2 possess an almost characteristic of

the second kind.

Since Ω and Δ may have different values subject to the restrictions imposed upon them in the hypothesis, it follows that genera of properly primitive ternary quadratic forms containing at least two classes of forms

$$f = ax^2 + \Omega q^2y^2 + cz^2 + 2\Omega r'yz + 2sxz$$

exist, having an almost characteristic of the second kind.

4. Lemma 3.3. If $(\gamma, pq\Omega\Delta) = 1$; if $\gamma \equiv 1, 3 \pmod 8$, when $p \equiv 5q, 7q \pmod 8$, and $\gamma \equiv 5, 7 \pmod 8$, when $p \equiv q, 3q \pmod 8$; and if $\Delta'' \geq 64$, then in case 7 of Lemma 1.3, where $p \equiv 1, 3 \pmod 8$, f represents primitively no $\Omega\gamma^2$.

If $f = \Omega\gamma^2$ has a primitive solution (x,y,z), then it follows as in Lemma 1.2 that u, v, and z can have no common prime factor other than p and q.

If $q \equiv 5 \pmod 8$, q does not divide z since $f = \Omega\gamma^2$, $z \equiv 0 \pmod q$ implies from

(28) $$\Omega\gamma^2 = p^2x^2 + \Omega q^2y^2 + cz^2 + 2\Omega r'yz + 2sxz$$

that

(29) $$\gamma^2 \equiv \Omega_2 p^2x^2 \pmod q$$

where

(30) $$\Omega\Omega_2 \equiv 1 \pmod q.$$

But

$$\left(\frac{\Omega_2 p^2x^2}{q}\right) = \left(\frac{\Omega^2\Omega_2}{q}\right) = \left(\frac{\Omega}{q}\right) = \left(\frac{2}{q}\right) = -1,$$

since $q \equiv 5 \pmod 8$. Hence q does not divide z, if $q \equiv 5 \pmod 8$.

From (4) and (5) with $a = p^2$ and $f = \Omega\gamma^2$, there is obtained

$$(31) \qquad \Omega^2 p^2 q^2 \gamma^2 = \Omega q^2 u^2 + p^2 \Omega^2 v_1^2 + \Omega^2 \Delta z^2, \quad v = \Omega v_1.$$

Hence (31) implies that u contains Ω as a factor. Therefore let $u = \Omega u_1$. Then

$$(32) \qquad p^2 q^2 \gamma^2 = \Omega q^2 u_1^2 + p^2 v_1^2 + \Delta z^2$$

and

$$(33) \qquad \Omega q^2 u_1^2 + p^2 v_1^2 = (pq\gamma)^2 - \Delta z^2$$

so that

$$(34) \qquad \Omega (qu_1)^2 + (pv_1)^2 = \rho\sigma$$

where

$$(35) \qquad \begin{aligned} \rho &= pq\gamma + \sqrt{\Delta}\, z \\ \sigma &= pq\gamma - \sqrt{\Delta}\, z. \end{aligned}$$

Take $\theta = \rho$ or σ. It follows as in Lemma 2.2 that $\theta > 0$, and $\theta \equiv 5, 7 \pmod 8$.

Assume there exist integers u_1, v_1, and z that satisfy (34). Then u_1, v_1, and θ must satisfy

$$(36) \qquad p^2 v_1^2 \equiv -\Omega q^2 u_1^2 \pmod \theta.$$

But

$$\left(\frac{-\Omega q^2 u_1^2}{\theta}\right) = \left(\frac{-\Omega}{\theta}\right) = \left(\frac{-2}{\theta}\right) = -1,$$

since $\theta \equiv 5, 7 \pmod 8$. Hence (36) has no primitive solution if $(-\Omega q^2 u_1^2, \theta) = 1$. Since $(\Omega, \theta) = 1$ and $\left(\frac{-\Omega}{\theta}\right) = -1$, there

exists an odd prime factor q_θ prime to Ω and dividing θ to an odd power such that $\left(\frac{-\Omega}{q_\theta}\right) = -1$. Therefore $\left(\frac{-2}{q_\theta}\right) = -1$ and $q_\theta \equiv 5, 7 \pmod 8$. Hence by assumption

(37) $$p^2v_1^2 \equiv -\Omega(qu_1)^2 \pmod{q_\theta}$$

has a solution with $(\Omega, q_\theta) = 1$ and $(-\Omega(qu_1)^2, q_\theta) > 1$. Therefore q_θ divides qu_1 and hence q_θ divides pv_1 so that (22) holds when u and p_θ are replaced by u_1 and q_θ respectively. By (35) with $q \equiv 5 \pmod 8$, $q_\theta \neq q$ since it has been shown that $z \not\equiv 0 \pmod q$. When $q \equiv 1 \pmod 8$, $q_\theta \neq q$ since $q_\theta \equiv 5, 7 \pmod 8$. Also $q_\theta \neq p$, since $p \equiv 1$, $3 \pmod 8$, so that (23) holds when u and p_θ are replaced by u_1 and q_θ respectively. Therefore $(q_\theta, z) = 1$, since by (5) q_θ would divide x and y which contradicts $(x,y,z) = 1$.

If q_θ divided both ρ and σ it would divide $pq\gamma$ and $\Omega\Delta z$ which has just been found to be impossible. Hence q_θ occurs to an even power in the left member of (34) and to an odd power in the right member, which is impossible. Hence (36) has no primitive solution and Lemma 3.3 is true.

5. Theorem 2.3. If Ω' and Δ' are odd squares, Ω'' is an odd power of 2 and Δ'' is an even power of 2 such that $\Delta'' \geq 64$, if p and q are distinct primes and prime to d, with $p \equiv 1, 3 \pmod 8$ and $q \equiv 1 \pmod 4$, and if each of the characters with respect to Ω and the characters with respect

to 4 and 8, when they exist, have the value one, then there exist genera of properly primitive positive ternary quadratic forms $f = p^2x^2 + \Omega q^2y^2 + cz^2 + 2\Omega r'yz + 2sxz$, with properly primitive reciprocals, containing at least two classes with an almost characteristic of the third kind.

For f of Lemma 1.3 define $f_1 = f$ when $b = \Omega q^2$, $p \equiv 5q, 7q \pmod 8$ and $f_2 = f$ when $b = \Omega q^2$, $p \equiv q, 3q \pmod 8$. For f_1 and f_2 take $a = p^2$, $p \equiv 1, 3 \pmod 8$, $t = 0$, and then s, r', and c are determined as in Lemma 1.3.

From Lemma 1.3 it is seen that $\left(\frac{F}{q_\Delta}\right) = \left(\frac{C}{q_\Delta}\right) = 1$ depends only upon Δ so that f_1 and f_2 belong to the same genus of forms. By Lemma 3.3 f_1 represents primitively no $\Omega\gamma^2$ where $\gamma \equiv 1, 3 \pmod 8$ and f_2 represents primitively no $\Omega\gamma^2$ where $\gamma \equiv 5, 7 \pmod 8$. Therefore the genus containing f_1 and f_2 contains at least two classes since f_1 and f_2 do not belong to the same class for there exists no odd square multiple of Ω represented primitively by both f_1 and f_2. Since Ω and Δ may have different values subject to the restrictions imposed upon them in the hypothesis, it follows that genera of properly primitive positive ternary quadratic forms, with properly primitive reciprocals, containing at least two classes of forms having an almost characteristic of the third kind exist, where $b = \Omega q^2$, $r = \Omega r'$, $t = 0$, and $P_{\Omega\text{-}\Delta} = 1$.

6. Example 1.3. Let $f = p^2x^2 + \Omega q^2y^2 + cz^2 + 2\Omega r'yz + 2sxz$, $\Omega = 8$ and $\Delta = 8$. Let $p = 5$ and $q = 3$. Then $s = 2$, $r' = 2$ and $c = 4$. Therefore

$$f_1 = 25x^2 + 72y^2 + 4z^2 + 32yz + 4xz.$$

The transformation

$$\begin{pmatrix} 0 & 0 & -1 \\ 0 & 1 & -1 \\ 1 & -4 & 4 \end{pmatrix}$$

reduces f_1 to

$$f_3 = 4X^2 + 8Y^2 + 17Z^2 - 2XZ.$$

Hence by Lemma 3.2 the class of forms represented by f_3 represents primitively no γ^2 where $\gamma \equiv 1, 3 \pmod 8$. Therefore this class possesses an almost characteristic of the second kind.

Now take $p = 17$. Then $s = 5$, $r' = 1$ and $c = 1$. Therefore

$$f_2 = 289x^2 + 72y^2 + z^2 + 16yz + 10xz.$$

The transformation

$$\begin{pmatrix} 0 & 0 & -1 \\ 0 & -1 & -5 \\ -1 & 8 & 45 \end{pmatrix}$$

reduces f_2 to

$$f_4 = X^2 + 8Y^2 + 64Z^2$$

Hence by Lemma 3.2 the class of forms represented by f_4 represents primitively no γ^2 where $\gamma \equiv 7 \pmod 8$.

Therefore this class possesses an almost characteristic of the second kind.

The characters exist with respect to 4 and 8:

$$\left(\frac{-1}{25}\right) = \left(\frac{-1}{289}\right) = 1,$$

$$\left(\frac{2}{25}\right) = \left(\frac{2}{289}\right) = 1,$$

$$\left(\frac{-1}{25\cdot 9}\right) = \left(\frac{-1}{289\cdot 9}\right) = 1,$$

$$\left(\frac{2}{25\cdot 9}\right) = \left(\frac{2}{289\cdot 9}\right) = 1.$$

Hence these two classes of forms are in the same genus.

Example 2.3. Let $f = p^2x^2 + \Omega q^2y^2 + oz^2 + 2\Omega r'yz + 2sxz$, $\Omega = 2$, and $\Delta = 64$. Take $p = 3$ and $q = 5$. Then $s = 1$, $r' = 2$, and $o = 1$. Therefore $f_1 = 9x^2 + 50y^2 + z^2 + 8yz + 2xz$.

The transformation

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 8 & -4 \\ 0 & -4 & 34 \end{pmatrix}$$

reduces f_1 to

$$f_3 = X^2 + 8Y^2 + 34Z^2 - 8YZ.$$

Hence by Lemma 3.2 the class of forms represented by f_3 represents primitively no $2\gamma^2$, where $\gamma \equiv 1, 3 \pmod 8$. Therefore f_3 possesses an almost characteristic of the third kind.

Now take $q = 17$. Then $s = 4$, $r' = 5$, and $c = 2$. Therefore $f_2 = 9x^2 + 578y^2 + 2z^2 + 20yz + 8xz$.

The transformation

$$\begin{pmatrix} 1 & 0 & -20 \\ 0 & 0 & -1 \\ -2 & 1 & 45 \end{pmatrix}$$

reduces f_2 to

$$f_4 = X^2 + 2Y^2 + 128Z^2.$$

Hence by Lemma 3.2 the class of forms represented by f_4 represents primitively no $2\gamma^2$ when $\gamma \equiv 5, 7 \pmod 8$. Therefore f_4 possesses an almost characteristic of the third kind.

The characters with respect to 4 and 8 which exist are:

$$\left(\frac{-1}{25\cdot 9}\right) = \left(\frac{-1}{289\cdot 9}\right) = 1,$$

$$\left(\frac{2}{25\cdot 9}\right) = \left(\frac{2}{289\cdot 9}\right) = 1.$$

Hence the two classes of forms are in the same genus.

CHAPTER IV

THE FORM $f = ax^2 + 2\Omega q^2y^2 + cz^2 + 2\Omega r'yz + 2sxz$ WHERE THE INVARIANTS Ω AND Δ CONTAIN AT LEAST ONE COMMON ODD PRIME FACTOR TO AN ODD POWER.

1. Lemma 1.4. Let $\Omega' = \Omega_1^2 P_{\Omega\Delta}$ and $\Delta' = \Delta_1^2 P_{\Omega\Delta}$ where Ω_1^2 and Δ_1^2 are the largest squares dividing Ω' and Δ' respectively. Then $P_{\Omega\Delta}$ is the common factor of Ω' and Δ' which is an odd prime or a product of distinct odd primes. Let $\Delta'' = 1$ and let Ω'' be an odd power of 2 such that $\Omega'' \geq 32$. Let p and q be distinct odd primes and prime to the determinant d, with $p \equiv 1 \pmod 4$. Let $\left(\frac{q}{P_{\Omega\Delta}}\right) = \left(\frac{-P_{\Omega\Delta}}{q}\right) = 1$. Let the characters with respect to Ω and the characters with respect to 4 and 8, when they exist, have the value one. Then a properly primitive positive ternary quadratic form

$$f = ax^2 + 2\Omega q^2y^2 + cz^2 + 2\Omega r'yz + 2sxz$$

exists, having a properly primitive reciprocal F, in case $P_{\Omega\Delta} \equiv 1 \pmod 4$, and an improperly primitive reciprocal F, in case $P_{\Omega\Delta} \equiv 3 \pmod 4$. Moreover $\left(\frac{F}{q_\Delta}\right) = \left(\frac{2}{q_\Delta}\right)$ for each odd prime factor q_Δ of Δ.

Take $a = 1$ or p^2, $b = 2\Omega q^2$, $r = \Omega r'$ and $t = 0$. The characters $\left(\frac{-1}{a}\right) = 1 = \left(\frac{2}{a}\right)$, and $\left(\frac{a}{p_\Omega}\right) = 1$ for each odd prime factor p_Ω of Ω. From the definition of a cofactor of an element of d, $C = 2aq^2$ so that $\left(\frac{F}{q_\Delta}\right) = \left(\frac{2}{q_\Delta}\right)$.

From the expression for the determinant d of f there is obtained

(38) $$aA - 2q^2s^2 = \Omega\Delta$$

where

(39) $$A = 2q^2c - \Omega r'^2.$$

Hence A is even so that

(40) $$q^2s^2 = -\frac{\Omega}{2}\Delta + \frac{aA}{2}.$$

Therefore with $a = p^2$,

(41) $$s^2 \equiv -\frac{\Omega}{2}\Delta q_1 \pmod{p^2}$$

where

(42) $$q^2q_1 \equiv 1 \pmod{p^2}.$$

The character

$$\left(\frac{-\frac{\Omega}{2}\Delta q_1}{p}\right) = \left(\frac{-\Omega^{2}p_{1\Omega\Delta}\frac{\Omega''}{2}\Delta^{2}p_{1\Omega\Delta}\,q^2q_1}{p}\right) = \left(\frac{-1}{p}\right) = 1,$$

since $p \equiv 1 \pmod 4$ and $\frac{\Omega''}{2}$ is an even power of 2. Hence (41) has a solution when s is odd. The value of $\frac{A}{2}$ is given by (40). From (40) $1 \equiv 0 + \frac{A}{2} \pmod 8$. Hence A is not a multiple of 4.

From (39) there is obtained

(43) $$r'^2 \equiv -\frac{A}{2}\,\Omega_2 \pmod{q^2}$$

where

(44) $$\frac{\Omega}{2}\Omega_2 \equiv 1 \pmod{q^2}.$$

From (38) note that $\frac{aA}{2} \equiv \frac{\Omega}{2}\Delta \pmod{q^2}$.

Then

$$\left(\frac{-\frac{A}{2}\Omega_2}{q}\right) = \left(\frac{-a\frac{A}{2}\Omega_2}{q}\right) = \left(\frac{-\frac{\Omega\Delta}{2}\Omega_2}{q}\right) = \left(\frac{-\Delta}{q}\right) = \left(\frac{-\Delta_1^2 P_{\Omega\Delta}}{q}\right)$$

$$= \left(\frac{\mp P_{\Omega\Delta}}{q}\right) = 1.$$

Therefore the congruence (43) has a solution. Also the value of r' is taken odd. Then the value of c is given by (39).

Since $(a, 2\Omega q^2) = 1$, f is properly primitive, and since (2) holds f is positive.

It must be shown that Ω is the greatest common divisor of the cofactors of the elements of d. The cofactors are

(45)
$$\begin{aligned}
\Omega A &= 2\Omega q^2 c - \Omega^2 r'^2 \\
\Omega B &= ac - s^2 \\
\Omega C &= 2\Omega a q^2 \\
\Omega R &= -a\Omega r' \\
\Omega S &= -2\Omega q^2 s \\
\Omega T &= \Omega r' s.
\end{aligned}$$

It is seen that all the elements of the determinant of F are integers with the possible exception of B. From (40), the fact that s was taken odd, and $\Omega'' \geq 32$ it is seen that

(46)
$$\frac{A}{2} \equiv 1 \pmod{8}.$$

From (39) it follows that

(47)
$$\frac{A}{2} = q^2 c - \frac{\Omega}{2} r'^2.$$

By (47), (46) and $\Omega'' \geq 32$ it follows that $c \equiv 1 \pmod 8$. Hence $s^2 \equiv ac \pmod{\Omega''}$ has a solution for s so that $\Omega' B = \frac{ac - s^2}{\Omega''}$ is an integer for the given values of a and Ω'' and the predetermined values of c and s. Hence if B is rational and not an integer its denominator must divide Ω'. From the expression for the determinant of d

$$2\Omega^2 q^2 B - a\Omega^2 r'^2 = \Omega^2 \Delta$$

so that

$$2q^2 B = ar'^2 + \Delta \tag{48}$$

where

$$ar'^2 + \Delta$$

is an integer. Therefore the denominator of B must divide $2q^2$. But $(2q^2, \Omega') = 1$. Therefore B is an integer.

Suppose the coefficients of F have a common divisor σ. Then σ divides both $C = 2aq^2$ and $\Omega\Delta^2$. But $(2aq^2, \Omega\Delta^2) = 2$. Hence σ divides 2. But σ must divide $r = -ar'$ and R is odd since r' was taken odd. Therefore $\sigma = 1$ and F is primitive. Hence Ω is the greatest common divisor of the cofactors of the elements of d.

From (48) $2B \equiv 1 + P_{\Omega_\Delta} \pmod 4$. If $P_{\Omega_\Delta} \equiv 1 \pmod 4$, then $B \equiv 1 \pmod 2$, and F is properly primitive. If $P_{\Omega_\Delta} \equiv 3 \pmod 4$, then $B \equiv 0 \pmod 2$, and F is improperly primitive. Hence Lemma 1.4 is true.

2. Lemma 2.4. Let $\Omega' = \Omega_1^2 P_{\Omega\Delta}$ and $\Delta' = \Delta_1^2 P_{\Omega\Delta}$ where Ω_1^2 and Δ_1^2 are the largest squares dividing Ω' and Δ' respectively. Let p and q be distinct odd primes and prime to d with $p \equiv 1 \pmod 4$. Let $\left(\frac{q}{P_{\Omega\Delta}}\right) = \left(\frac{-P_{\Omega\Delta}}{q}\right) = 1$. Let Ω'' be an odd power of 2 greater than or equal to 128, and let $\Delta'' = 1$. Let γ be odd, positive, and prime to $pq\Omega\Delta$. Let $\left(\frac{\gamma}{P_{\Omega\Delta}}\right) = -\left(\frac{p}{P_{\Omega\Delta}}\right)$. Also when $P_{\Omega\Delta} \equiv 1 \pmod 4$, let $\gamma \equiv pq \pmod 8$. Then

$$f = p^2x^2 + 2\Omega q^2y^2 + cz^2 + 2\Omega r'yz + 2sxz$$

represents primitively no γ^2.

If $f = \gamma^2$ has a primitive solution (x,y,z), then u, v, and z can have no common prime factors except p and q. For suppose an odd prime g divides u, v, and z and is prime to p and q. Then by (5) g divides x, y, and z and the solution is not primitive. If 2 divides u, v, and z, then 2 divides x so that f is even. But $f = \gamma^2$ is odd.

If $\left(\frac{p}{P_{\Omega\Delta}}\right) = -1$, p does not divide z, since $f = \gamma^2$, $z \equiv 0 \pmod p$ implies

(49) $$\gamma^2 \equiv 2\Omega q^2y^2 \pmod p.$$

The character

$$\left(\frac{2\Omega q^2y^2}{p}\right) = \left(\frac{2\Omega_1^2 P_{\Omega\Delta}\Omega''}{p}\right) = \left(\frac{P_{\Omega\Delta}}{p}\right) = \left(\frac{p}{P_{\Omega\Delta}}\right) = -1.$$

From (4) there is obtained, with $a = p^2$ and $f = \gamma^2$,

$$p^2q^2\gamma^2 = (qu)^2 + \frac{\Omega}{2}(pv_1)^2 + \frac{\Omega}{2}\Delta z^2$$

so that

(50) $$(qu)^2 + \frac{\Omega}{2}(pv_1)^2 = (pq\gamma)^2 - \frac{\Omega}{2}\Delta z^2.$$

Then

(51) $$(qu)^2 + \frac{\Omega}{2}(pv_1)^2 = \rho\sigma$$

where $$\rho = pq\gamma + \sqrt{\frac{\Omega}{2}\Delta}\, z$$

(52) $$\sigma = pq\gamma - \sqrt{\frac{\Omega}{2}\Delta}\, z.$$

Take $\theta = \rho$ or σ. Then, as in Lemma 2.2, θ is greater than zero. Also from (52), $\theta \equiv 1 \pmod 8$ when $\gamma \equiv pq \pmod 8$.

Assume there exist integers u, v_1, and z that satisfy (51), then

(53) $$(qu)^2 \equiv -\frac{\Omega}{2}(pv_1)^2 \pmod \theta.$$

If $(\frac{\Omega}{2}(pv_1)^2, \theta) = 1$, then

$$\left(\frac{-\frac{\Omega}{2}(pv_1)^2}{\theta}\right) = \left(\frac{-\frac{\Omega}{2}}{\theta}\right) = \left(\frac{-\Omega_1^2 P_{\Omega\Delta}\frac{\Omega''}{2}}{\theta}\right) = \left(\frac{-P_{\Omega\Delta}}{\theta}\right) = \left(\frac{\theta}{P_{\Omega\Delta}}\right)$$

since either $P_{\Omega\Delta} \equiv 3 \pmod 4$ or $\theta \equiv 1 \pmod 8$.

From (52),

$$\left(\frac{\theta}{P_{\Omega\Delta}}\right) = \left(\frac{pq\gamma}{P_{\Omega\Delta}}\right) = \left(\frac{p\gamma}{P_{\Omega\Delta}}\right) = -1.$$

Hence there is no solution to the congruence (53).

Since $\left(\frac{-\frac{\Omega}{2}}{\theta}\right) = -1$ and $(\frac{\Omega}{2}, \theta) = 1$, there exists an odd prime factor p_θ prime to $\frac{\Omega}{2}$ and dividing θ to an odd power such that $\left(\frac{-\frac{\Omega}{2}}{p_\theta}\right) = -1$. Hence by assumption

(54) $$(qu)^2 \equiv -\frac{\Omega}{2}(pv_1)^2 \pmod{p_e}$$

has a solution when $\left(\frac{\Omega}{2}(pv_1)^2, p_e\right) > 1$. Hence p_e divides pv_1 and therefore must divide qu. From (52) it is seen that if $\left(\frac{p}{P_{\Omega\Delta}}\right) = -1$, then $p_e \neq p$ since it has been shown $z \not\equiv 0 \pmod{p}$. If $\left(\frac{p}{P_{\Omega\Delta}}\right) = 1$, $p_e \neq p$ since $\left(\frac{p_e}{P_{\Omega\Delta}}\right) = -1$. Also $p_e \neq q$ since $\left(\frac{q}{P_{\Omega\Delta}}\right) = 1$. Hence (23) holds.

By an argument similar to that in Lemma 2.2, it follows that $(p_e, z) = 1$, and that congruences (54) and (53) have no solutions. Hence there exists no primitive solution of $f = \gamma^2$. Therefore Lemma 2.4 is true.

3. Theorem 1.4. Let $\Omega' = \Omega_1^2 P_{\Omega\Delta}$ and $\Delta' = \Delta_1^2 P_{\Omega\Delta}$ where Ω_1^2 and Δ_1^2 are the largest squares dividing Ω' and Δ' respectively. Let $P_{\Omega\Delta}$, Ω'', Δ'', p, q, and γ have the properties as in Lemma 2.4. Let the characters with respect to Ω, and the characters with respect to 4 and 8, when they exist, have the value one. Then there exist genera of properly primitive positive ternary quadratic forms

$$f = p^2x^2 + 2\Omega q^2y^2 + cz^2 + 2\Omega r'yz + 2sxz$$

containing at least two classes of forms with an almost characteristic of the second kind, and whose reciprocal forms are properly or improperly primitive according as $P_{\Omega\Delta} \equiv 1$ or $3 \pmod 4$.

For f of Lemma 1.4 take $a = p^2$ and define $f_1 = f$, when $\left(\frac{p}{P_{\Omega\Delta}}\right) = -1$ and $f_2 = f$, when $\left(\frac{p}{P_{\Omega\Delta}}\right) = 1$. For f_1 and f_2

take $b = 2\Omega q^2$, $r = \Omega r'$, $t = 0$, and then determine s, r, and c as in Lemma 1.4. From Lemma 1.4 $\left(\frac{F}{q_\Delta}\right) = 1$ depends only upon Δ so that f_1 and f_2 belong to the same genus of forms. By Lemma 2.4, $f_1 \neq \gamma^2$, when $\left(\frac{\gamma}{P_{\Omega\Delta}}\right) = -\left(\frac{p}{P_{\Omega\Delta}}\right) = 1$, and $f_2 \neq \gamma^2$, when $\left(\frac{\gamma}{P_{\Omega\Delta}}\right) = -\left(\frac{p}{P_{\Omega\Delta}}\right) = -1$.

Therefore the genus containing f_1 and f_2, contains at least two classes since there is no odd square represented primitively by both f_1 and f_2.

Because Ω and Δ may have different values subject to the restrictions placed upon them in the hypothesis, it follows that genera of properly primitive positive ternary quadratic forms containing at least two classes of forms possessing an almost characteristic of the second kind exist, and whose reciprocal forms are properly or improperly primitive according as $P_{\Omega\Delta} \equiv 1$ or $3 \pmod 4$.

4. Example 1.4. Let $\Omega = 384$ and $\Delta = 3$. Then $P_{\Omega\Delta} = 3$. Take $p = 5$ and $q = 7$, then $\left(\frac{q}{P_{\Omega\Delta}}\right) = \left(\frac{-P_{\Omega\Delta}}{q}\right) = 1$, $\left(\frac{\gamma}{3}\right) = -\left(\frac{5}{3}\right) = 1$, and odd $\gamma \equiv 1 \pmod 3$. Hence $s = -1$, $A = 50$, $r' = -27$, and $c = 2857$. Therefore $f_1 = 25x^2 + 37632y^2 + 2857z^2 - 20736yz - 2xz$.

Now take $p = 13$, then $\left(\frac{\gamma}{3}\right) = -\left(\frac{13}{3}\right) = -1$ and odd $\gamma \equiv 2 \pmod 3$. Hence $s = 71$, $a = 2930$, $r' = -33$, and $c = 4297$. Therefore $f_2 = 169x^2 + 37632y^2 + 4297z^2 - 25344yz$

$+ 142xz.$

The characters

$$\left(\frac{-1}{25}\right) = \left(\frac{-1}{169}\right) = 1,$$

$$\left(\frac{2}{25}\right) = \left(\frac{2}{169}\right) = 1,$$

and $$\left(\frac{25}{3}\right) = 1.$$

It follows that f_1 and f_2 are in different classes but in the same genus and each possesses an almost characteristic of the second kind.

CHAPTER V

THE FORM $f = ax^2 + 2\Omega q^2y^2 + cz^2 + 2\Omega r'yz + 2sxz$ WHERE THE INVARIANTS Ω AND Δ CONTAIN NO COMMON ODD PRIME FACTOR TO AN ODD POWER

1. Lemma 1.5. Let Ω' and Δ' be odd squares, Ω'' be an even power of 2 such that $\Omega'' \geq 4$ and $\Delta'' = 2$. Let the characters with respect to Ω and the characters with respect to 4 and 8, when they exist, have the value one. Let p and q be distinct odd primes and prime to the determinant d with $p \equiv 1 \pmod 4$ and $q \equiv 1, 3 \pmod 8$. Then properly primitive forms f and F exist for which $a = 1$ or p^2 and $b = 2\Omega q^2$. Moreover $\left(\frac{F}{q_\Delta}\right) = \left(\frac{2}{q_\Delta}\right)$ for each odd prime factor q_Δ of Δ.

Take $a = 1$ or p^2, $b = 2\Omega q^2$, $r = \Omega r'$, and $t = 0$. The character $\left(\frac{a}{p_\Omega}\right) = 1$ for each odd prime factor p_Ω of Ω. Also $\left(\frac{-1}{a}\right) = 1 = \left(\frac{2}{a}\right)$. From the definition of a cofactor of an element of d, $C = 2aq^2$ so that $\left(\frac{F}{q_\Delta}\right) = \left(\frac{C}{q_\Delta}\right) = \left(\frac{2}{q_\Delta}\right)$.

From (41) the character $\left(\frac{\frac{\Omega\Delta q_1}{2}}{p}\right) = \left(\frac{-\frac{\Omega''\Delta''}{2}}{p}\right) = \left(\frac{-1}{p}\right)$ $= 1$, since $p \equiv 1 \pmod 4$. Hence (41) has a solution for s. Also s is taken odd. Then the value of $\frac{A}{2}$ is given by (40).

From (39) with $r' = 2r_1$ there is obtained

(55) $$r_1^2 \equiv - \frac{A}{2}\Omega_2 \pmod{q^2}$$

where

(56) $$2\Omega\Omega_2 \equiv 1 \pmod{q^2}.$$

From (40) $a\frac{A}{2} \equiv \frac{\Omega\Delta}{2} \pmod{q^2}$ so that with (56)

$$\left(\frac{-\frac{A}{2}\Omega_2}{q}\right) = \left(\frac{-\frac{A}{2}\frac{\Omega}{2}2\Omega\Omega_2}{q}\right) = \left(\frac{-\frac{A}{2}a\frac{\Omega}{2}}{q}\right) = \left(\frac{-\frac{\Omega^2}{4}\Delta}{q}\right) = \left(\frac{-\Delta}{q}\right)$$

$= \left(\frac{-\Delta''}{q}\right) = \left(\frac{-2}{q}\right) = 1$, since $q \equiv 1, 3 \pmod 8$. Therefore the congruence (55) has a solution and the value of c is given by (39).

Since $(a, 2\Omega q^2) = 1$, f is properly primitive, and since (2) holds, f is positive.

It must be shown that Ω is the greatest common divisor of the cofactors of the elements of d. By the expressions for these cofactors, namely:

(57)
$$\begin{aligned}
\Omega A &= 2\Omega q^2c^2 - 4\Omega^2 r_1^2 \\
\Omega B &= ac - s^2 \\
\Omega C &= 2a\Omega q^2 \\
\Omega R &= - 2\Omega a r_1 \\
\Omega S &= - 2\Omega q^2 s \\
\Omega T &= 2\Omega r_1 s,
\end{aligned}$$

it is seen that all the coefficients of F are integers with the possible exception of B. It is also seen that B must be

odd, if F is to be primitive.

From the expression for d there is obtained

(58) $$2q^2B = 4ar_1^2 + \Delta'\Delta'', \quad r' = 2r_1,$$

so that

(59) $$q^2B = 2ar_1^2 + \Delta'$$

is an odd integer. If B is not an integer its denominator must divide q^2. But $\Omega B = ac - s^2$, so that the denominator of B must divide Ω. Since $(q^2, \Omega) = 1$, B must be an integer. By (59) it is seen that B is also odd.

A common divisor σ of the coefficients of F must divide both $C = 2aq^2$ and $\Omega\Delta^2$. But $(2aq^2, \Omega\Delta^2) = 2$, so that σ divides 2. Also σ divides B an odd integer. Hence $\sigma = 1$ and F is primitive. Therefore Ω is the greatest common divisor of the cofactors of the elements of d. Since (A,B,C) is odd, F is properly primitive. Hence Oemma 1.5 is true.

2. Lemma 2.5. Let Ω' and Δ' be odd squares, Ω'' be an even power of 2 such that $\Omega'' \geq 64$ and let $\Delta'' = 2$. Let p and q be distinct primes and prime to d. Let $p \equiv 1 \pmod 4$ and $q \equiv 1, 3 \pmod 8$. Then

$$f = p^2x^2 + 2\Omega q^2y^2 + cz^2 + 2\Omega r'yz + 2sxz$$

represents primitively no γ^2 where γ is positive, prime to $pq\Omega\Delta$, and $\gamma \equiv 5pq$ or $7pq \pmod 8$.

If $f = \gamma^2$ has a primitive solution (x,y,z), then u, v, and z can have no common prime factors except p and q. This follows by an argument identical to that of Lemma 2.4.

If $p \equiv 5 \pmod 8$, p cannot divide z since $f = \gamma^2$, $z \equiv 0 \pmod p$ implies

(60) $$\gamma^2 \equiv 2\Omega q^2 y^2 \pmod p.$$

But $\left(\frac{2\Omega q^2 y^2}{p}\right) = \left(\frac{2}{p}\right) = -1$. Hence the congruence (60) has no solution and p does not divide z. From (4) and (5) with $a = p^2$ and $f = \gamma^2$

$$p^2 q^2 \gamma^2 = q^2 u^2 + 2\Omega p^2 v_2^2 + \frac{\Omega}{2}\Delta z^2, \quad v = 2\Omega v_2$$

and hence

(61) $$(qu)^2 + 2\Omega(pv_2)^2 = (pq\gamma)^2 - \frac{\Omega}{2}\Delta z^2$$

so that

(62) $$(qu)^2 + 2\Omega(pv_2)^2 = \rho\sigma$$

where

(63) $$\rho = pq\gamma + \sqrt{\frac{\Omega}{2}\Delta}z$$
$$\sigma = pq\gamma - \sqrt{\frac{\Omega}{2}\Delta}z.$$

Take $\theta = \rho$ or σ. Now σ and ρ have the same sign since the left member of (62) is positive. But $\rho + \sigma = 2pq\gamma > 0$, therefore $\theta > 0$.

Assume there exist integers u, v_1, and z that satisfy (61). Then u, v_1, and θ must satisfy

(64) $$(qu)^2 \equiv -2\Omega(pv_2)^2 \pmod \theta.$$

By (63) $\theta \equiv pq\gamma \equiv 5$ or $7 \pmod 8$, so that

$$\left(\frac{-2\Omega(pv_2)^2}{\theta}\right) = \left(\frac{-2\Omega}{\theta}\right) = \left(\frac{-2}{\theta}\right) = -1.$$

Hence (64) has no solution if $(2\Omega(pv_2)^2, \theta) = 1$, with

$(2\Omega,\theta) = 1$ and $\left(\frac{-2\Omega}{\theta}\right) = -1$. Hence there exists an odd prime factor p_θ, prime to 2Ω, and dividing θ to an odd power such that $\left(\frac{-2\Omega}{p_\theta}\right) = -1$. Therefore $\left(\frac{-2}{p_\theta}\right) = -1$ and $p_\theta \equiv 5, 7 \pmod 8$. Hence by assumption

(65) $$(qu)^2 \equiv -2\Omega(pv_2)^2 \pmod{p_\theta}$$

has a solution with $(2\Omega,p_\theta) = 1$ and $(-2\Omega(pv_2)^2,p_\theta) > 1$. Therefore p_θ divides pv_2 amd hence p_θ divides qu so that (22) holds. With $p \equiv 5 \pmod 8$, $p_\theta \neq p$ since it has been shown $z \not\equiv 0 \pmod p$, and with $p \equiv 1 \pmod 8$, $p_\theta \neq p$, since $p_\theta \equiv 5, 7 \pmod 8$. Also $p_\theta \neq q$ since $q \equiv 1, 3 \pmod 8$. Therefore (23) holds. Hence $(p_\theta,z) = 1$ and the congruences (64) and (65) have no solution by an argument similar to that of Lemma 2.2. Therefore Lemma 2.5 is true.

3. Theorem 1.5. Let Ω' and Δ' be odd squares, Ω'' be an even power of 2 such that $\Omega'' > 64$ and $\Delta'' = 2$. Let the characters with respect to Ω and the characters with respect to 4 and 8, when they exist, have the value one. Let p and q be distinct odd primes and prime to the determinant d, and let $p \equiv 1 \pmod 4$ and $q \equiv 1, 3 \pmod 8$. Also let Ω and Δ contain no common odd prime factor to an odd power. Then there exist genera of properly primitive positive ternary quadratic forms

$$f = p^2x^2 + 2\Omega q^2y^2 + cz^2 + 2\Omega r'yz + 2sxz,$$

with properly primitive reciprocals F, containing at least

two classes of forms with an almost characteristic of the second kind.

For f of Lemma 1.5 define $f_1 = f$ when $a = p^2$ and $p \equiv 1 \pmod 8$ and $f_2 = f$ when $a = p^2$ and $p \equiv 5 \pmod 8$. For f_1 and f_2, take $b = 2\Omega q^2$, $r = 2\Omega r_1$, $t = 0$, and then determine s, r, and c as in Lemma 1.5. From Lemma 1.5 it is seen that $\left(\frac{F}{q_\Delta}\right) = \left(\frac{C}{q_\Delta}\right) = \left(\frac{2}{q_\Delta}\right)$ depends only upon Δ so that f_1 and f_2 belong to the same genus of forms. By Lemma 2.5 f_1 represents primitively no γ^2 where $\gamma \equiv 1, 3 \pmod 8$ and f_2 represents primitively no γ^2 where $\gamma \equiv 5, 7 \pmod 8$. Therefore the genus containing f_1 and f_2 contains at least two classes since f_1 and f_2 do not belong to the same class, for there exists no odd square represented primitively by both f_1 and f_2. Since Ω and Δ may have different values subject to the restrictions imposed upon them in the hypothesis, it follows that genera of properly primitive positive ternary quadratic forms containing at least two classes exist possessing an almost characteristic of the second kind.

4. Example 1.5. Let $\Omega = 64$ and $\Delta = 2$. Take $p = 17$ and $2 = 3$. Then $s = 5$, $A = 2$, $r' = 2r_1 = 22$, and $c = 1721$. Therefore

$$f_1 = 289x^2 + 1152y^2 + 1721z^2 + 5632yz + 102xz$$

represents primitively no γ^2 where $\gamma \equiv 1, 3 \pmod 8$. Hence f_1 possesses an almost characteristic of the second kind.

The characters

$$\left(\frac{-1}{289}\right) = \left(\frac{-1}{25}\right) = 1,$$

$$\left(\frac{2}{289}\right) = \left(\frac{2}{25}\right) = 1.$$

Hence f_1 and f_2 are in the same genus of forms.

CHAPTER VI

THE FORM $f = ax^2 + 4\Omega q^2y^2 + cz^2 + 2\Omega r'yz + 2sxz$ WHERE THE INVARIANTS Ω AND Δ CONTAIN AT LEAST ONE COMMON ODD PRIME FACTOR TO AN ODD POWER.

1. Lemma 1.6. Let $\Omega' = \Omega_1^2 P_{\Omega\Delta}$ and $\Delta' = \Delta_1^2 P_{\Omega\Delta}$ where Ω_1^2 and Δ_1^2 are the largest squares dividing Ω' and Δ' respectively. Then $P_{\Omega\Delta}$ is the common factor of Ω' and Δ' which is an odd prime or a product of distinct odd primes. Let the characters with respect to 4 and 8, when they exist, have the value one. Let p and q be distinct odd primes and prime to the determinant d. Let $\left(\frac{q}{P_{\Omega\Delta}}\right) = 1$ and let $P_{\Omega\Delta}$, Ω'', Δ'', p and q have the properties as listed for the particular cases in Table 4 on page 51.

Then a properly primitive positive ternary quadratic form f exists for which $a = 1$ or p^2, $b = 4\Omega q^2$, $r = \Omega r'$, and $t = 0$. Further, the reciprocal form F is properly primitive in all cases except in cases 1 and 5 with $P_{\Omega\Delta} \equiv 7 \pmod 8$ when it is improperly primitive. Moreover $\left(\frac{F}{q_\Delta}\right) = 1$ for each odd prime factor q_Δ of Δ.

To show the existence of f take

$$a = 1 \text{ or } p^2,\ b = 4\Omega q^2,\ r = \Omega r',\ \text{and } t = 0.$$

Also

$$\left(\frac{-1}{a}\right) = 1 = \left(\frac{2}{a}\right) \text{ and } \left(\frac{a}{p_\Omega}\right) = 1$$

for each odd prime factor of p_Ω of Ω.

TABLE 4

Case	$P_{\Omega\Delta} \equiv$ (mod 4)	Power of 2 Ω''	Power of 2 Δ''	Ω''	Δ''	$\Omega''\Delta''$	$p \equiv$	$q \equiv$
1	3	even	even	≥ 64	1	≥ 64	1 (mod 4)	. . .
2	3	even	even	$= 1$	≥ 16	≥ 16	1 (mod 4)	. . .
3	3	even	even	≥ 4	≥ 16	≥ 64	1 (mod 4)	. . .
4	3	odd	odd	≥ 2	≥ 8	≥ 16	1 (mod 4)	1,7(mod 8)
5	3	odd	even	≥ 32	$= 1$	≥ 32	1,3(mod 8)	. . .
6	3	odd	even	≥ 2	≥ 16	≥ 32	1,3(mod 8)	. . .
7	3	even	odd	≥ 4	≥ 8	≥ 32	1,3(mod 8)	1,7(mod 8)
8	3	even	odd	$= 1$	≥ 8	≥ 8	1,3(mod 8)	1,7(mod 8)
9	1	even	even	$= 1$	≥ 16	≥ 16	1 (mod 4)	1 (mod 4)
10	1	even	even	≥ 4	≥ 16	≥ 64	1 (mod 4)	1 (mod 4)
11	1	odd	odd	≥ 2	≥ 8	≥ 16	1 (mod 4)	1 (mod 8)
12	1	odd	even	≥ 2	≥ 16	≥ 32	1,3(mod 8)	1 (mod 4)
13	1	even	odd	≥ 4	≥ 8	≥ 32	1,3(mod 8)	1 (mod 8)
14	1	even	odd	$= 1$	≥ 8	≥ 8	1,3(mod 8)	1 (mod 8)

From the definition of a cofactor of an element of d, $C = 4aq^2$ so that $\left(\frac{F}{q_\Delta}\right) = \left(\frac{C}{q_\Delta}\right) = 1$. From the expression for d there is obtained

(66) $$aA - 4q^2s^2 = \Omega\Delta$$

where

(67) $$A = 4q^2c - \Omega r'^2.$$

From (66) it follows that

(68) $$s^2 \equiv -\frac{\Omega\Delta}{4}q_1 \pmod{p^2}$$

where

(69) $$q^2q_1 \equiv 1 \pmod{p^2}.$$

The quadratic character

$$\left(\frac{-\frac{\Omega\Delta}{4}q_1}{p}\right) = \left(\frac{-\frac{\Omega\Delta q^2 q_1}{4}}{p}\right) = \left(\frac{-\frac{\Omega\Delta}{4}}{p}\right) = \left(\frac{-\Omega_1^{2p_{\Omega-\Delta}}\Delta_1^{2p_{\Omega-\Delta}}\frac{''\Delta''}{4}}{p}\right)$$

$$= \left(\frac{-\frac{\Omega''\Delta''}{4}}{p}\right).$$

For cases 1, 2, 3, 4, 9, 10, and 11, $\left(\frac{-\frac{\Omega''\Delta''}{4}}{p}\right) = \left(\frac{-1}{p}\right) = 1$,

since $p \equiv 1 \pmod 4$; and for cases 5, 6, 7, 8, 12, 13, and 14, $\left(\frac{-\frac{\Omega''\Delta''}{4}}{p}\right) = \left(\frac{-2}{p}\right) = 1$, since $p \equiv 1, 3 \pmod 8$. Hence the congruence (68) has an integral solution for s, and the integral value of A given by (66) is a multiple of 4, but not of 8, when s is chosen odd.

Consider cases 1 and 5. It is seen from (67) that

(70) $$\frac{A}{4} = q^2c - \frac{\Omega}{4}r'^2.$$

It follows from (70) that

(71) $$r'^2 \equiv -\frac{A}{4}\Omega_2 \pmod{q^2}$$

where

(72) $$\frac{\Omega\Omega_2}{4} \equiv 1 \pmod{q^2}$$

has a solution for odd r' since, noting from (66) that

(73) $$\frac{aA}{4} \equiv \frac{\Omega\Delta}{4} \pmod{q^2},$$

$$\left(\frac{-\frac{A}{4}\Omega_2}{q}\right) = \left(\frac{-\frac{aA}{4}\Omega_2}{q}\right) = \left(\frac{-\frac{\Omega\Delta}{4}\Omega_2}{q}\right) = \left(\frac{-\Delta}{q}\right) = \left(\frac{-\Delta_1^2 P_{\Omega\Delta}\,\Delta''}{q}\right)$$

$$= \left(\frac{-P_{\Omega\Delta}}{q}\right) = \left(\frac{q}{P_{\Omega\Delta}}\right) = 1.$$

Hence the value of c, given by (70), is odd.

Consider all cases, not 1 and 5. Let

(74) $$r' = 2r_1.$$

Then from (70) it follows that

(75) $$\frac{A}{4} = q^2 c - \Omega r_1^2$$

and

(76) $$r_1^2 \equiv -\frac{A}{4}\Omega_3 \pmod{q^2}$$

where

(77) $$\Omega\Omega_3 \equiv 1 \pmod{q^2}.$$

From (73) and (77) the quadratic character

$$\left(\frac{-\frac{A}{4}\Omega_3}{q}\right) = \left(\frac{-\frac{aA}{4}\Omega_3}{q}\right) = \left(\frac{-\frac{\Omega\Delta}{4}\Omega_3}{q}\right) = \left(\frac{-\frac{\Delta}{4}}{q}\right) = \left(\frac{-P_{\Omega\Delta}}{q}\right)\left(\frac{\frac{\Delta''}{4}}{q}\right)$$

$$= \left(\frac{q}{P_{\Omega\Delta}}\right)\left(\frac{\frac{\Delta''}{4}}{q}\right) = \left(\frac{\frac{\Delta''}{4}}{q}\right) = 1.$$

Hence (76) has a solution for r_1, which is taken odd, and the integral value of c is given by (75).

For all cases 1 - 14, $(a, 4\Omega q^2) = 1$, f is properly primitive, and since (2) holds f is positive.

It must be shown that Ω is the greatest common divisor of the cofactors of the elements of d. By the expression for the cofactors of a, c, r, s, and t with $t = 0$, it is obvious that in all cases, A, C, R, S, and T are integers. Also ΩB is given

(78) $$\Omega B = ac - s^2.$$

For cases 1 and 5, it is seen from (66) that $\frac{A}{4} \equiv 1 \pmod 8$. From (70) it follows that $1 \equiv c \pmod 8$. Hence $ac \equiv 1 \pmod 8$ so that $s^2 \equiv ac \pmod{\Omega''}$ has a solution. Therefore $\Omega' B = \frac{ac - s^2}{\Omega''}$ is an integer for the given values of a and Ω'' and the predetermined values c and s. Hence if B is rational and not an integer its denominator must divide Ω'. From d there is obtained

(79) $$4\Omega^2 q^2 B - a\Omega^2 r'^2 = \Omega^2 \Delta$$

so that

(80) $$4q^2 B = ar'^2 + \Delta$$

where $ar'^2 + \Delta$ is an integer. Therefore the denominator of B must divide $4q^2$. But $(4q^2, \Omega') = 1$. Hence B is an integer for the cases 1 and 5.

In all other cases there is obtained from d

(81) $$4\Omega^2 q^2 B - 4a\Omega^2 r_1^2 = \Omega^2 \Delta$$

so that

(82) $$q^2 B = ar_1^2 + \frac{\Delta}{4}$$

where $ar_1^2 + \frac{\Delta}{4}$ is an integer. Hence the denominator of B divides q^2. But the denominator of B divides Ω since $\Omega B = ac - s^2$ and $ac - s^2$ is an integer. Therefore, since $(q^2, \Omega) = 1$, B is an integer for the remaining cases, not 1 and 5.

For cases 1 and 5 it follows from (80) that

(83) $$4B = 1 + P_{\Omega\Delta} \pmod 8$$

and hence B is odd or even according as $P_{\Omega\Delta} \equiv 3$ or $7 \pmod 8$. Also $R = -ar'$ is odd since r' was taken odd. From (83) and the expressions for the cofactors of a and c, (A,B,C) is odd or even according as $P_{\Omega\Delta} \equiv 3$ or $7 \pmod 8$ and hence it follows that F is properly or improperly primitive according as $P_{\Omega\Delta} \equiv 3$ or $7 \pmod 8$, provided F is primitive. Let σ be a common divisor of the elements of the determinant of F. Then σ divides both $C = 4aq^2$ and $\Omega\Delta^2$. But $(4aq^2, \Omega\Delta^2) = 4$, so that σ divides 4. Also σ divides $R = -ar'$ an odd integer. Hence $\sigma = 1$ and F is primitive.

For all cases, not 1 and 5, it is seen from (82) that B is odd so that F is properly primitive provided it is primitive. Let σ be a common divisor of the elements of the determinant of F.
Then σ divides both $C = 4aq^2$ and $\Omega\Delta^2$. But $(4aq^2, \Omega\Delta^2) = 4$, so that σ divides 4. Also σ divides B an odd integer. Hence $\sigma = 1$ and F is primitive.

In all cases, not 1 and 5, of Table 4 where $\Delta'' > 16$, $B \equiv 1 \pmod 8$. Hence $\left(\frac{-1}{B}\right) = 1 = \left(\frac{2}{B}\right)$.

In cases 2, 3, 6, 9, 10, and 12 with $\Delta'' = 16$, $B \equiv 5 \pmod 8$. Hence $\left(\frac{-1}{B}\right) = 1$ and $\left(\frac{2}{B}\right) = -1$.

In cases 4, 7, and 8 with $\Delta'' = 8$, $B \equiv 7 \pmod 8$. Hence $\left(\frac{-1}{B}\right) = -1$ and $\left(\frac{2}{B}\right) = 1$.

In cases 11, 13, and 14 with $\Delta'' = 8$, $B \equiv 3 \pmod 8$. Hence $\left(\frac{-1}{B}\right) = -1 = \left(\frac{2}{B}\right)$.

Therefore Lemma 1.6 is true.

2. Lemma 2.6. Let $\Omega' = \Omega_1^2 P_{\Omega\Delta}$ and $\Delta' = \Delta_1^2 P_{\Omega\Delta}$ where Ω_1^2 and Δ_1^2 are the largest squares dividing Ω' and Δ' respectively. Let $P_{\Omega\Delta}$, Ω'', Δ'', p, and q be defined as in Lemma 1.6. Let $\left(\frac{q}{P_{\Omega\Delta}}\right) = 1$ and let odd $\gamma > 0$ have the values as listed for the particular cases in

TABLE 5[1]

Case	$P_{\Omega\Delta} \equiv$ (4)	Power of 2 Ω''	Δ''	Ω''	Δ''	$\Omega''\Delta''$	p $\equiv$	q $\equiv$	$\gamma \equiv$ (8)
1	3	even	even	≥ 64	$= 1$	≥ 64	1 (4)	. . .	. .
2	3	even	even	$=$	≥ 16	≥ 16	1 (4)	. . .	. .
3	3	even	even	≥ 4	≥ 16	≥ 64	1 (4)	. . .	. .
4	3	odd	odd	≥ 2	≥ 8	≥ 256	1 (8)	1,7 (8)	pq
5	1	even	even	$= 1$	≥ 64	≥ 64	1 (4)	1 (4)	pq
6	1	even	even	≥ 4	≥ 16	≥ 64	1 (4)	1 (4)	pq
7	1	odd	Odd	≥ 2	≥ 8	≥ 256	1 (8)	1 (8)	pq

[1]In this table (4) and (8) denote (mod 4) and (mod 8) respectively.

Let $\left(\frac{\gamma}{P_{\Omega\Delta}}\right) = -\left(\frac{p}{P_{\Omega\Delta}}\right)$ and $(\gamma, pq\Omega\Delta) = 1$. Then f of Lemma 1.6 represents primitively no γ^2.

Cases 1-4 of Table 5 are identical with cases 1-4 of Table 4 respectively, except for additional restrictions upon $\Omega''\Delta''$ and upon p in case 4. Also cases 5-7 of Table 5 are identical with cases 9-11 of Table 4 respectively, except for further restrictions upon Δ'', $\Omega''\Delta''$, and upon p in case 7.

Now in case 1 of Lemma 1.6, r' was taken odd, whereas in cases 2-4, and cases 9-11 of Lemma 1.6, $r' = 2r_1$ as is seen by (74).

If $f = \gamma^2$ has a primitive solution (x,y,z), then the fact that u, v, and z can have no common prime factors except p and q follows from an argument similar to that given in Lemma 2.4.

If $\left(\frac{p}{P_{\Omega\Delta}}\right) = -1$, P does not divide z since $f = \gamma^2$, $z \equiv 0 \pmod p$ implies

(84) $$\gamma^2 \equiv 4\Omega q^2 y^2 \pmod p.$$

The quadratic character

$$\left(\frac{4\Omega q^2 y^2}{p}\right) = \left(\frac{\Omega}{p}\right) = \left(\frac{\Omega_1^2 P_{\Omega\Delta}\Omega''}{p}\right) = \left(\frac{P_{\Omega\Delta}}{p}\right)\left(\frac{\Omega''}{p}\right).$$

For cases 1, 2, 3, 5, and 6 of Table 5,

$$\left(\frac{P_{\Omega\Delta}}{p}\right)\left(\frac{\Omega''}{p}\right) = \left(\frac{p}{P_{\Omega\Delta}}\right) = -1$$

since $p \equiv 1 \pmod 4$. For cases 4 and 7,

$$\left(\frac{p_{\Omega\Delta}}{p}\right)\left(\frac{\Omega''}{p}\right) = \left(\frac{p}{p_{\Omega\Delta}}\right)\left(\frac{2}{p}\right) = \left(\frac{p}{p_{\Omega\Delta}}\right) = -1,$$

since $p \equiv 1 \pmod 8$. Hence p does not divide z, if $\left(\frac{p}{p_{\Omega\Delta}}\right) = -1$. Consider case 1. From (4) and (5) with $a = p^2$ and $f = \gamma^2$ there is obtained

$$p^2q^2\gamma^2 = q^2u^2 + \frac{\Omega}{4}p^2v_1^2 + \frac{\Omega}{4}\Delta z^2, \quad v = \Omega v_1$$

so that

(85) $$(qu)^2 + \frac{\Omega}{4}(pv_1)^2 = (pq\gamma)^2 - \frac{\Omega\Delta_z 2}{4}$$

and

(86) $$(qu)^2 + \frac{\Omega}{4}(pv_1)^2 = \rho\sigma$$

where

(87) $$\rho = pq\gamma + \frac{1}{2}\sqrt{\Omega\Delta}z.$$

$$\sigma = pq\gamma - \frac{1}{2}\sqrt{\Omega\Delta}z.$$

Consider all other cases of Table 5. From (4) and (5) with $a = p^2$ and $f = \gamma^2$ there is obtained

(88) $$p^2q^2\gamma^2 = q^2u^2 + \Omega p^2v_2^2 + \frac{\Omega\Delta_z 2}{4},$$

where

$$v = \Omega v_1 = 2\Omega v_2.$$

Hence

(89) $$(qu)^2 + \Omega(pv_2)^2 = (pq\gamma)^2 - \frac{\Omega\Delta_z 2}{4}.$$

Then

(90) $$(qu)^2 + \Omega(pv_2)^2 = \rho\sigma$$

where

(91) $$\rho = pq\gamma + \frac{1}{2}\sqrt{\Omega\Delta}\,z$$

$$\sigma = pq\gamma - \frac{1}{2}\sqrt{\Omega\Delta}\,z.$$

In all cases take $\theta = \rho$ or σ. ρ and σ must have the same sign since the left members of (86) and (90) are positive. From (87) and (91), $\rho + \sigma = 2pq\gamma$. Hence $\theta > 0$ in all cases.

Assume there exist two sets of integers (u, v_1, z) and (u, v_2, z) which satisfy (85) and (89) respectively. Then (u, v_1, θ) and (u, v_2, θ) must satisfy

(92) $$(qu)^2 \equiv -\frac{\Omega}{4}(pv_1)^2 \pmod{\theta}$$

and

(93) $$(qu)^2 \equiv -\Omega(pv_2)^2 \pmod{\theta}$$

respectively.

Consider case 1. From (92) the quadratic character

$$\left(\frac{\frac{-\Omega}{4}(pv_1)^2}{\theta}\right) = \left(\frac{-\Omega_1^2 P_{\Omega\Delta}\frac{\Omega''}{4}}{\theta}\right) = \left(\frac{-P_{\Omega\Delta}}{\theta}\right) = \left(\frac{\theta}{P_{\Omega\Delta}}\right) = \left(\frac{pq\gamma}{P_{\Omega\Delta}}\right)$$

$$= -1.$$

Hence (92) has no solution with $\left(\frac{\Omega}{4}(pv_1)^2, \theta\right) = 1$. Since

$\left(\frac{\frac{-\Omega}{4}}{\theta}\right) = -1$, there exists an odd prime factor p_θ, prime to

$\frac{\Omega}{4}$, and dividing θ to an odd power, such that $\left(\frac{-\frac{\Omega}{4}}{\theta}\right) = -1.$

Hence by assumption

(94) $$(qu)^2 \equiv -\frac{\Omega}{4}(pv_1)^2 \pmod{p_\theta}$$

has a solution when $(\frac{\Omega}{4}(pv_1)^2, p_\theta) > 1$, and since p_θ does not divide $\frac{\Omega}{4}$ it must divide pv_1 and hence must divide qu. Therefore (22) holds. When $\left(\frac{p}{p_{\Omega\Delta}}\right) = -1$, $p_\theta \neq p$ since it has been shown $z \not\equiv 0 \pmod p$. When $\left(\frac{p}{p_{\Omega\Delta}}\right) = 1$, $p_\theta \neq p$ since $\left(\frac{p_\theta}{p_{\Omega\Delta}}\right) = -1$. Also $p_\theta \neq q$ since $\left(\frac{q}{p_{\Omega\Delta}}\right) = 1$. Hence (23) holds. Therefore $(p_\theta, z) = 1$ and the congruences (94) and (92) have no solution by an argument similar to that given in Lemma 2.2. Hence there is no primitive solution of $f = \gamma^2$ for case 1.

Consider cases 2, 3, 5, and 6. From (93) the quadratic character

$$\left(\frac{-\Omega(pv_2)^2}{\theta}\right) = \left(\frac{-\Omega}{\theta}\right) = \left(\frac{-\Omega_1^2 p_{\Omega\Delta}\Omega''}{\theta}\right) = \left(\frac{-p_{\Omega\Delta}}{\theta}\right)\left(\frac{\Omega''}{\theta}\right)$$

$$= \left(\frac{\theta}{p_{\Omega\Delta}}\right)\left(\frac{\Omega''}{\theta}\right) = \left(\frac{pq\gamma}{p_{\Omega\Delta}}\right) = \left(\frac{p\gamma}{p_{\Omega\Delta}}\right) = -1.$$

Hence the congruence (93) has no solution if $(-\Omega(pv_2)^2, \theta) = 1$. Since $\left(\frac{-\Omega}{\theta}\right) = -1$, there exists an odd prime factor p_θ prime to Ω and dividing θ to an odd power such that $\left(\frac{-\Omega}{\theta}\right) = -1$. Hence by assumption

(95) $$(qu)^2 \equiv - \Omega(pv_2)^2 \pmod{p_\theta}$$

has a solution when $(\Omega(pv_2)^2, p_\theta) > 1$, and since p_θ does not divide Ω it must divide pv_2 and therefore divides qu. Hence with v_1 replaced by v_2, (23) holds, and an argument identical with that given for case 1 of this Lemma shows there is no primitive solution of $f = \gamma^2$ for cases 2, 3, 5 and 6.

Consider cases 4 and 7. From (93) the quadratic character

$$\left(\frac{-\Omega(pv_2)^2}{\theta}\right) = \left(\frac{-\Omega_1^2 P_{\Omega\Delta}\,\Omega''}{\theta}\right) = \left(\frac{-P_{\Omega\Delta}}{\theta}\right)\left(\frac{2}{\theta}\right) = \left(\frac{-P_{\Omega\Delta}}{\theta}\right),$$

since $\theta \equiv 1 \pmod 8$. Therefore

$$\left(\frac{-P_{\Omega\Delta}}{\theta}\right) = \left(\frac{\theta}{P_{\Omega\Delta}}\right) = \left(\frac{pq\gamma}{P_{\Omega\Delta}}\right) = \left(\frac{p\gamma}{P_{\Omega\Delta}}\right) = -1.$$

Hence (93) has no solution if $(\Omega(pv_2)^2, \theta) = 1$. By the precise argument used in cases 2, 3, 5 and 6 $f = \gamma^2$ has no primitive solution in these cases. Hence Lemma 2.6 is true.

3. Theorem 1.6. Let $\Omega' = \Omega_1^2 P_{\Omega\Delta}$ and $\Delta' = \Delta_1^2 P_{\Omega\Delta}$ where Ω_1^2 and Δ_1^2 are the largest squares dividing Ω' and Δ' respectively. Let $P_{\Omega\Delta}$, Ω'', Δ'', p, q, and odd $\gamma > 0$ have the properties as listed for the particular cases in Table 5. Let p and q be distinct odd primes and prime to the determinant d, with $\left(\frac{q}{P_{\Omega\Delta}}\right) = 1$. Let the characters with

respect to Ω and the characters with respect to 4 and 8, when they exist, have the value one. Then there exist genera of properly primitive positive ternary quadratic forms $f = ax^2 + 4\Omega q^2y^2 + cz^2 + 2\Omega r'yz + 2sxz$, with primitive reciprocal forms F, containing at least two classes of forms possessing an almost characteristic of the second kind. Finally F is properly primitive except for case 1 in Table 5 with $P_{\Omega\Delta} \equiv 7 \pmod 8$, when it is improperly primitive.

For f of Lemma 1.6 take $a = p^2$ and define $f_1 = f$ when $\left(\frac{p}{P_{\Omega\Delta}}\right) = -1$ and $f_2 = f$, when $\left(\frac{p}{P_{\Omega\Delta}}\right) = 1$. For f_1 and f_2 take $b = 4\Omega q^2$, $r = \Omega r'$ and $t = 0$ and then s, r', and c are determined as in Lemma 1.6. From Lemma 1.6 $\left(\frac{F}{q_\Delta}\right) = 1$ depends only upon Δ so that f_1 and f_2 belong to the same genus. By Lemma 2.6 $f_1 \neq \gamma^2$ where $\left(\frac{\gamma}{P_{\Omega\Delta}}\right) = -\left(\frac{p}{P_{\Omega\Delta}}\right) = 1$, and $f_2 \neq \gamma^2$ where $\left(\frac{\gamma}{P_{\Omega\Delta}}\right) = -\left(\frac{p}{P_{\Omega\Delta}}\right) = -1$. Therefore the genus containing f_1 and f_2 contains at least two classes of forms since there is no odd square represented primitively by both f_1 and f_2.

Because Ω and Δ may have different values subject to the restrictions imposed upon them in the hypothesis, it follows that genera of properly primitive positive ternary quadratic forms $f = ax^2 + 4\Omega q^2y^2 + cz^2 + 2\Omega r'yz + 2sxz$ exist, containing at least two classes of forms possessing

an almost characteristic of the second kind, where $P_{\Omega\Delta} > 1$. Further, the reciprocal forms F are properly primitive in all cases except for case 1 in Table 5 with $P_{\Omega\Delta} \equiv 7 \pmod 8$ when it is improperly primitive.

4. Example 1.6. Let $\Omega = 3$ and $\Delta = 48$, then $P_{\Omega\Delta} = 3$. Take $p = 5$ and $q = 7$, then $\left(\frac{p}{P_{\Omega\Delta}}\right) = \left(\frac{5}{3}\right) = -1$,

$$\left(\frac{q}{P_{\Omega\Delta}}\right) = \left(\frac{7}{3}\right) = 1 = \left(\frac{-3}{7}\right) = \left(\frac{-P_{\Omega\Delta}}{q}\right), \text{ and } \left(\frac{\gamma}{3}\right) = -\left(\frac{5}{3}\right) = 1.$$

Hence odd $\gamma \equiv 1 \pmod 3$. It is found that $s = 31$, A $A = 7540$, $r' = 2r_1 = 10$, and $c = 40$. Hence

$$f_1 = 25x^2 + 588y^2 + 40z^2 + 60yz + 62xz$$

represents primitively no γ^2 where odd $\gamma \equiv 1 \pmod 3$, and f_1 possesses an almost characteristic of the second kind.

Now take $p' = 13$, then $\left(\frac{13}{3}\right) = 1$ and $\left(\frac{\gamma}{3}\right) = -1$.

Therefore odd $\gamma \equiv 2 \pmod 3$. It follows that $s = 229$, $A = 60820$, $r' = 2r_1 = 34$, and $c = 328$. Hence

$$f_2 = 169x^2 + 588y^2 + 328z^2 + 204yz + 458xz$$

represents primitively no γ^2 where odd $\gamma \equiv 2 \pmod 3$ and f_2 possesses an almost characteristic of the second kind.

The characters

$$\left(\frac{-1}{B}\right) = \left(\frac{-1}{13}\right) = \left(\frac{-1}{997}\right) = 1,$$

$$\left(\frac{2}{B}\right) = \left(\frac{2}{13}\right) = \left(\frac{2}{997}\right) = -1,$$

$$\left(\frac{p}{P_{\Omega}}\right) = \left(\frac{25}{3}\right) = \left(\frac{169}{3}\right) = 1.$$

Hence f_1 and f_2 are in the same genus of forms.

CHAPTER VII

THE FORM $f = ax^2 + 4\Omega q^2y^2 + cz^2 + 2\Omega r'yz + 2sxz$ WHERE THE INVARIANTS Ω AND Δ CONTAIN NO COMMON ODD PRIME FACTOR TO AN ODD POWER

1. Lemma 1.7. Let Ω' and Δ' be odd squares. Let the characters with respect to Ω and the characters with respect to 4 and 8, when they exist, have the value one. Let $P_{\Omega\Delta} = 1$ and let p and q be distinct odd primes and prime to d, and let Ω'', Δ'', p, and q have the properties listed for the particular cases in

TABLE 6

Case	Power of 2						
	Ω''	Δ''	Ω''	Δ''	$\Omega''\Delta''$	$p \equiv$	$q \equiv$
1	even	even	$= 1$	≥ 16	≥ 16	1 (mod 4)	1 (mod 4)
2	even	even	≥ 4	≥ 16	≥ 64	1 (mod 4)	1 (mod 4)
3	odd	odd	≥ 2	≥ 8	≥ 16	1 (mod 4)	1,3(mod 8)
4	odd	even	≥ 2	≥ 16	≥ 32	1,3(mod 8)	1 (mod 4)
5	even	odd	$= 1$	≥ 8	≥ 8	1,3(mod 8)	1,3(mod 8)
6	even	odd	≥ 4	≥ 8	≥ 32	1,3(mod 8)	1,3(mod 8)

Then properly primitive positive ternary quadratic forms $f = ax^2 + 4\Omega q^2y^2 + cz^2 + 2\Omega r'yz + 2sxz$ and F exist. Moreover $\left(\frac{F}{q_\Delta}\right) = 1$ for each odd prime factor q_Δ of Δ.

To show the existence of the form f take $a = 1$ or p^2. Then the character $\left(\frac{a}{p_\Omega}\right) = 1$ for each odd prime factor p_Ω of Ω. Also $\left(\frac{-1}{a}\right) = 1 = \left(\frac{2}{a}\right)$.

From the definition of a cofactor of an element of d, $C = 4aq^2$ so that $\left(\frac{F}{q_\Delta}\right) = \left(\frac{C}{q_\Delta}\right) = 1$.

From the expression for the determinant d, (66) and (67) are obtained. Then (68) has a solution, because

$$\left(\frac{\frac{-\Omega\Delta q_1}{4}}{p}\right) = \left(\frac{\frac{-\Omega\Delta}{4}}{p}\right) = \left(\frac{\frac{-\Omega'\Omega''\Delta'\Delta''}{4}}{p}\right) = \left(\frac{\frac{-\Omega''\Delta''}{4}}{p}\right) = 1, \text{ since}$$

$\left(\frac{\frac{-\Omega''\Delta''}{4}}{p}\right) = \left(\frac{-1}{p}\right)$ for cases 1, 2, and 3 of Table 6, and

$\left(\frac{\frac{-\Omega''\Delta''}{4}}{p}\right) = \left(\frac{-2}{p}\right)$ for cases 4, 5, and 6 of the table.

From the expression for the determinant d of f (80) is obtained. Hence r' must be even since Δ'' is even. Therefore use (74), and (67) becomes (75) whence the congruences (76) and (77) must hold. The quadratic character

$\left(\frac{-\frac{A}{4}\Omega_3}{q}\right) = \left(\frac{-\frac{aA}{4}\Omega_3}{q}\right) = \left(\frac{-\frac{\Omega\Delta}{4}\Omega_3}{q}\right) = \left(\frac{\frac{-\Delta''}{4}}{q}\right)$, which is equal to $\left(\frac{-1}{q}\right)$ for cases 1, 2, and 4; and to $\left(\frac{-2}{q}\right)$ for cases 3, 5, and 6. Hence in all cases $\left(\frac{\frac{-\Delta''}{4}}{q}\right) = 1$, since $q \equiv 1 \pmod 4$ for

cases 1, 2, and 4 and $q \equiv 1, 3 \pmod 8$ for cases 3, 5, and 6. Hence the congruence (76) has a solution in r_1 which is taken odd. The integral value of c is given by (75) where c is even in cases 1 and 5 but is odd for the remaining cases.

Since a is odd and $(a, 4\Omega q^2) = 1$, f is properly primitive, and since (2) holds, f is positive.

By an argument similar to that used in cases, not 1 and 5, of Lemma 1.6 it may be shown that B is an odd integer, and that Ω is the greatest common divisor of the cofactors of the elements of d.

In all cases of Table 6 with $\Delta'' > 16$, $B \equiv 1 \pmod 8$.

Hence $\left(\dfrac{-1}{B}\right) = 1 = \left(\dfrac{2}{B}\right)$.

In cases 1, 2, and 4 with $\Delta'' = 16$, $B \equiv 5 \pmod 8$.

Hence $\left(\dfrac{-1}{B}\right) = 1$ and $\left(\dfrac{2}{B}\right) = -1$.

In cases 3, 5, and 6 with $\Delta'' < 16$, $B \equiv 3 \pmod 8$.

Hence $\left(\dfrac{-1}{B}\right) = -1 = \left(\dfrac{2}{B}\right)$. Therefore Lemma 1.7 is true.

2. Lemma 2.7. Let Ω' and Δ' be odd squares, Ω'' and Δ'' each be odd powers of 2, and $\Omega''\Delta'' \geq 256$. Let $p \equiv 1 \pmod 4$ and $q \equiv 1, 3 \pmod 8$ be distinct primes and prime to d. Then f of Lemma 1.7 represents primitively no γ^2 where γ is positive, prime to $pq\Omega\Delta$, and $\gamma \equiv 5pq$ or $7pq \pmod 8$;

If $f = \gamma^2$ has a primitive solution (x,y,z), then u, v, and z can have no common prime factors except p and q,

since by (5) such a prime factor would divide x, y, and z and the solution would not be primitive.

If $p \equiv 5 \pmod 8$, p cannot divide z since $f = \gamma^2$, $z \equiv 0 \pmod p$ implies (84). But $\left(\frac{4\Omega q^2 y^2}{p}\right) = \left(\frac{\Omega}{p}\right) = \left(\frac{2}{p}\right) = -1$.

Hence (84) has no solution and p does not divide z.

From (4) and (5) with $a = p^2$, $f = \gamma^2$ and $v = 2\Omega v_2$ there is obtained (88) from which (89), (90) and (91) are obtained. Take $\theta = \rho$ or σ. Then by an argument identical with that in Lemma 2.6, θ is found to be > 0. Also by (91) and the fact that $\gamma \equiv 5pq$ or $7pq \pmod 8$ it is found that $\theta \equiv 5$ or $7 \pmod 8$.

Assume there exist integers u, v_2, and z that satisfy (89). Then u, v_2, and θ must satisfy (93). But $\left(\frac{-\Omega(pv_2)^2}{\theta}\right) = \left(\frac{-\Omega}{\theta}\right) = \left(\frac{-\Omega''}{\theta}\right) = \left(\frac{-2}{\theta}\right) = -1$, since $\theta \equiv 5$ or $7 \pmod 8$. Hence (93) has no solution if $(\Omega(pv_2)^2, \theta) = 1$. But $(\Omega, \theta) = 1$ and $\left(\frac{-\Omega}{\theta}\right) = -1$. Hence there exists an odd prime factor p_θ prime to Ω and dividing θ to an odd power such that $\left(\frac{-\Omega}{p_\theta}\right) = -1$. Therefore $\left(\frac{-2}{p_\theta}\right) = -1$ and $p_\theta \equiv 5, 7 \pmod 8$. Hence by assumption (95) has a solution when $(\Omega(pv_2)^2, p_\theta) > 1$, and since p_θ does not divide Ω it must divide pv_2 and therefore divides qu. Hence, with v_1 replaced by v_2, (23) holds, and an argument similar to that

given in Lemma 2.2 shows there is no primitive solution to $f = \gamma^2$. Hence Lemma 2.7 is true.

3. Theorem 1.7. If Ω' and Δ' are odd squares, Ω'' and Δ'' are each odd powers of 2 with $\Omega''\Delta'' \geq 256$, Ω and Δ contain no common odd prime factor to an odd power, and each of the characters with respect to Ω and the characters with respect to 4 and 8, when they exist, have the value one, then there exist genera of properly primitive positive ternary quadratic forms

$$f = p^2x^2 + 4\Omega q^2y^2 + cz^2 + 2\Omega r'yz + 2sxz,$$

with properly primitive reciprocals F, containing at least two classes of forms with an almost characteristic of the second kind.

For f of Lemma 1.7 define $f_1 = f$, when $p \equiv 1 \pmod 8$, and $f_2 = f$, when $p \equiv 5 \pmod 8$.

For f_1 and f_2, take $b = 4\Omega q^2$, $r = \Omega r' = 2\Omega r_1$, $t = 0$, and then s, r, and c are determined as in Lemma 1.7. From Lemma 1.7 it is seen that $\left(\frac{F}{q_\Delta}\right) = \left(\frac{C}{q_\Delta}\right) = 1$ depends only upon Δ so that f_1 and f_2 belong to the same genus of forms. By Lemma 2.7 f_1 represents primitively no γ^2 where $\gamma \equiv 5$ or $7 \pmod 8$ and f_2 represents primitively no γ^2 where $\gamma \equiv 1$ or $3 \pmod 8$. Therefore the genus containing f_1 and f_2 contains at least two classes since f_1 and f_2 belong to different classes for there exists no odd square represented

primitively by both f_1 and f_2. Since Ω and Δ may have different values subject to the restrictions imposed upon them in the hypothesis, it follows that genera of properly primitive positive ternary quadratic forms containing at least two classes of forms with an almost characteristic of the second kind exist.

4. Example 1.7. Let $\Omega = 2$ and $\Delta = 128$. Take $p = 17$ and $q = 3$. Then $s = 5$, $A = 4$, $r' = 2r_1 = 22$, and $c = 27$. Hence

$$f_1 = 289x^2 + 72y^2 + 27z^2 + 88yz + 10xz$$

represents primitively no γ^2 where $\gamma \equiv 5, 7 \pmod 8$ so that f_1 possesses an almost characteristic of the second kind.

Now take $p = 5$. Then $s = 27$, $A = 1060$, $r' = 2r_1 = 10$, and $c = 35$. Hence

$$f_2 = 25x^2 + 72y^2 + 35z^2 + 40yz + 54xz$$

represents primitively no γ^2 where $\gamma \equiv 1, 3 \pmod 8$. So that f_2 possesses an almost characteristic of the second kind.

The characters

$$\left(\frac{-1}{B}\right) = \left(\frac{-1}{3889}\right) = \left(\frac{-1}{73}\right) = 1,$$

$$\left(\frac{2}{B}\right) = \left(\frac{2}{3889}\right) = \left(\frac{2}{73}\right) = 1.$$

Hence f_1 and f_2 are in the same genus of forms.

CHAPTER VIII

THE FORM $f = 2p^2x^2 + 2\Omega q^2y^2 + cz^2 + 2\Omega r'yz + 2sxz$ WHERE THE INVARIANTS Ω AND Δ CONTAIN NO COMMON ODD PRIME FACTOR TO AN ODD POWER

1. Lemma 1.8. Let Ω' and Δ' be odd squares where each prime factor of Ω' is congruent to 1 or 7 modulo 8, and let $\Omega'' = 1$ and Δ'' be an even power of 2 with $\Delta'' \geq 16$. Let the characters with respect to Ω and the characters with respect to 4 and 8, when they exist, have the value 1. Let p and q be distinct odd primes and prime to the determinant d with both p and q congruent to 1 or 3 modulo 8, and let $P_{\Omega\Delta} = 1$. Then properly primitive forms

$$f = 2p^2x^2 + 2\Omega q^2y^2 + cz^2 + 2\Omega r'yz + 2sxz,$$

where $(c, 2pq\Omega) = 1$, exist, with properly primitive reciprocals F. Moreover $\left(\frac{F}{q_\Delta}\right) = 1$ for each odd prime factor q_Δ of Δ.

Take $a = 2p^2$, $b = 2\Omega q^2$, $r = \Omega r'$, and $t = 0$. The character $\left(\frac{2p^2}{p_\Omega}\right) = 1$ since each prime factor of Ω' is congruent to 1 or 7 modulo 8. From the definition of a cofactor of an element of d, $C = 4p^2q^2$ so that $\left(\frac{F}{q_\Delta}\right) = \left(\frac{C}{q_\Delta}\right) = 1$ for each odd prime factor q_Δ of Δ.

From the expression for the determinant d of f there is obtained

(96) $$p^2A - q^2s^2 = \frac{\Omega\Delta}{2}$$

where

(97) $$A = 2q^2c - \Omega r'^2.$$

Hence A is odd if s and r' are chosen odd. From (96) it follows that

(98) $$s^2 \equiv - \frac{\Omega\Delta}{2}q_1 \pmod{p^2}$$

where

(99) $$q^2q_1 \equiv 1 \qquad \pmod{p^2}$$

so that

$$\left(\frac{-\frac{\Omega\Delta}{2}q_1}{p}\right) = \left(\frac{-\frac{\Omega\Delta}{2}}{p}\right) = \left(\frac{-\frac{\Delta''}{2}}{p}\right) = \left(\frac{-2}{p}\right) = 1,$$

since $p \equiv 1, 3 \pmod 8$. Hence (98) has a solution for s. Also s is chosen odd, and the integral value of A is given by (96).

From (97) it follows that

(100) $$r'^2 \equiv - A\,\Omega_2 \pmod{2q^2}$$

where

(101) $$\Omega\Omega_2 \equiv 1 \qquad \pmod{2q^2}.$$

Note from (96) that $p^2A \equiv \frac{\Omega\Delta}{2} \pmod{q^2}$. Then the quadratic character

$$\left(\frac{-A\Omega_2}{q}\right) = \left(\frac{-p^2A\Omega_2}{q}\right) = \left(\frac{-\Omega\frac{\Delta}{2}\Omega_2}{q}\right) = \left(\frac{-2}{q}\right) = 1,$$

since $q \equiv 1, 3 \pmod 8$. Hence (100) has a solution for r'.

Also r' is chosen odd, and the integral value of c is given by (97). Moreover $(c, 2p^2, 2\Omega q^2) = 1$ so that f is properly primitive. Also (2) holds. Hence f is positive.

It must be shown that Ω is the greatest common divisor of the cofactors of the elements of d. By the expressions for these cofactors, namely

$$
(102) \qquad \begin{aligned}
\Omega A &= 2\Omega q^2 c - \Omega 2 r'^2 \\
\Omega B &= 2p^2 c - s^2 \\
\Omega C &= 4\Omega p^2 q^2 \\
\Omega R &= -2\Omega p^2 r' \\
\Omega S &= -2\Omega q^2 s \\
\Omega T &= \Omega r' s,
\end{aligned}
$$

it is seen that each element of the determinant of F is an integer with the possible exception of B.

From the expression for d, there is obtained

$$
(103) \qquad q^2 B = p^2 r'^2 + \frac{\Delta}{2}.
$$

If B is rational and not an integer its denominator must divide q^2. Also the denominator of B must divide Ω since $\Omega B = 2p^2c - s^2$ and $2p^2c - s^2$ is an integer. But $(q^2, \Omega) = 1$. Hence B is an integer. Further, it is seen from (103) that B must be odd. Hence F is properly primitive, provided it is primitive. A common divisor σ of the elements of the determinant of F must divide both $C = 4p^2q^2$ and $\Omega\Delta^2$. Hence σ must divide 4. But σ must divide B an odd integer. Therefore $\sigma = 1$ and F is primitive. Hence Ω is the greatest

common divisor of the cofactors of the elements of d.

From (103) it is seen that $B \equiv 1 \pmod 8$. Hence $\left(\frac{-1}{B}\right) = 1 = \left(\frac{2}{B}\right)$, and Lemma 1.8 is true.

2. Lemma 2.8. Let Ω' and Δ' be odd squares where each prime factor of Ω' is congruent to 1 or 7 (mod 8). Let $\Omega'' = 1$ and $\Delta'' \geq 16$ be an even power of 2. Let p and q be distinct primes and prime to d, with p and $q \equiv 1$ or 3 (mod 8). Let γ be positive and prime to $pq\Omega\Delta$. Let $\gamma \equiv 1 \pmod 4$ when $p \equiv q$ or $3q \pmod 8$ respectively according as $\Delta'' = 16$ or $\Delta'' > 16$. Let $\gamma \equiv 3 \pmod 4$ when $p \equiv 3q$ or $q \pmod 8$ respectively according as $\Delta'' = 16$ or $\Delta'' > 16$. Then

$$f = 2p^2x^2 + 2\Omega q^2y^2 + cz^2 + 2\Omega r'yz + 2sxz$$

of Lemma 1.8 represents primitively no γ^2.

If $f = \gamma^2$ has a primitive solution (x,y,z) with z odd, then from (4) and (5), u, v, z can have no common prime factors except p and q by an argument similar to that for Lemma 2.2

If $p \equiv 3 \pmod 8$, p does not divide z since $f = \gamma^2$, $z \equiv 0 \pmod p$ implies that

(104) $$\gamma^2 \equiv 2\Omega q^2y^2 \pmod p.$$

But $\left(\frac{2\Omega q^2y^2}{p}\right) = \left(\frac{2}{p}\right) = -1$, since $p \equiv 3 \pmod 8$ by hypothesis. Hence the congruence (104) has no solution and p does not divide z.

From (4) and (5) it follows that

(105) $$(qu)^2 + \Omega(pv_1)^2 = 2\left[(pq\gamma)^2 - \frac{\Omega\Delta}{4}z^2\right]$$

so that

(106) $$(qu)^2 + \Omega(pv_1)^2 = 2\rho\sigma$$

where

(107) $$\rho = pq + \frac{1}{2}\sqrt{\Omega\Delta}\,z$$
$$\sigma = pq - \frac{1}{2}\sqrt{\Omega\Delta}\,z,$$

Take $\theta = \rho$ or σ. Then by an argument similar to that for Lemma 2.2, θ is found to be > 0. Note that if $f = \gamma^2$, z must be odd. Observe by (107) that $\theta \equiv 3 \pmod 4$.

Assume there exists integers u, v_1, and z that satisfy (105). Then u, v_1, and θ must satisfy.

(108) $$(qu)^2 \equiv -\Omega(pv_1)^2 \pmod{\theta}.$$

Consider the case where $(\Omega(pv_1)^2, \theta) = 1$.

$$\left(\frac{-\Omega(pv_1)^2}{\theta}\right) = \left(\frac{-\Omega}{\theta}\right) = \left(-\frac{1}{\theta}\right) = -1, \text{ since } \theta \equiv 3 \pmod 4.$$

Hence there is no solution if $(\Omega(pv_1)^2, \theta) = 1$. Therefore consider the case where $(\Omega(pv_1)^2, \theta) > 1$. Since $\left(\frac{-\Omega}{\theta}\right) = -1$, there exists an odd prime factor p_θ, prime to Ω, and dividing θ to an odd power such that $\left(\frac{-\Omega}{p_\theta}\right) = \left(\frac{-1}{p_\theta}\right) = -1$.

Hence $p_\theta \equiv 3 \pmod 4$. Therefore if

(109) $$(qu)^2 \equiv -\Omega(pv_1)^2 \pmod{p_\theta}$$

has a solution p_θ divides pv_1 and divides qu. Therefore (22) holds.

If $p \equiv 3 \pmod 8$, $p_e \neq p$ since $z \not\equiv 0 \pmod p$ as shown. If $p \equiv 1 \pmod 8$, $p_e \neq p$ since $p_e \equiv 3, 7 \pmod 8$. Also $p_e \neq q \equiv 1 \pmod 8$.

If $q \equiv 3 \pmod 8$, q does not divide z since $f = \gamma 2$, $z \equiv 0 \pmod q$ implies

(110) $$\gamma^2 \equiv 2p^2x^2 \pmod q.$$

But $\left(\frac{2p^2x^2}{q}\right) = \left(\frac{2}{q}\right) = -1$ since $q \equiv 3 \pmod 8$. Hence $p_e \neq q \equiv 3 \pmod 8$ since $z \not\equiv 0 \pmod q$ has just been shown. Therefore (23) holds. Hence $(p_e, z) = 1$ since otherwise p_e would divide x and y by (5). Therefore the congruences (108) and (109) have no solution and Lemma 2.8 is true.

3. Theorem 1.8. Let Ω' and Δ' be odd squares with each prime factor of Ω' congruent to 1 or 7 (mod 8), $\Omega'' = 1$, and $\Delta'' \geq 16$ be an even power of 2. Let p and q be distinct primes each $\equiv 1, 3 \pmod 8$ and $(pq, \Omega^2\Delta) = 1$. Let $(\gamma, pq\Omega\Delta) = 1$. Let $\gamma \equiv 1 \pmod 4$ when $p \equiv q$ or $3q \pmod 8$ respectively according as $\Delta'' = 16$ or $\Delta'' > 16$. Let $\gamma \equiv 3 \pmod 4$ when $p \equiv 3q$ or $q \pmod 8$ respectively according as $\Delta'' = 16$ or $\Delta'' > 16$. Let the characters with respect to Ω and the characters with respect to 4 and 8, when they exist, have the value one. Then there exist genera of properly primitive positive ternary quadratic forms

$$f = 2p^2x^2 + 2\Omega q^2y^2 + cz^2 + 2\Omega r'yz + 2sxz,$$

where $(c, 2pq\Omega) = 1$, with properly primitive reciprocals F,

containing at least two classes of forms possessing an almost characteristic of the second kind.

For f of Lemma 1.8 define $f_1 = f$ when $\gamma \equiv pq \pmod 4$ or $3\,pq \pmod 4$ according as $\Delta'' = 16$ or $\Delta'' > 16$. Define $f_2 = f$ when $\gamma \equiv 3\,pq \pmod 4$ or $pq \pmod 4$ according as $\Delta'' = 16$ or $\Delta'' > 16$.

For f_1 and f_2 take $b = 2\Omega q^2$, $a = 2p^2$, $r \equiv \Omega r'$, $t = 0$, and then s, r, and c are determined as in Lemma 1.8. From Lemma 1.8 it is seen that $\left(\frac{F}{q_\Delta}\right)$ depends only on Δ so that f_1 and f_2 belong to the same genus of forms. By Lemma 2.8, f_1 represents primitively no γ^2 where $\gamma \equiv 1 \pmod 4$ and f_2 represents primitively no γ^2 where $\gamma \equiv 3 \pmod 4$. Therefore the genus containing f_1 and f_2 contains at least two classes for there exists no odd square represented primitively by both f_1 and f_2.

Since Ω and Δ may have different values subject to the restrictions imposed upon them in the hypothesis, it follows that genera of properly primitive positive ternary quadratic forms of at least two classes of forms with an almost characteristic of the second kind exist.

4. Example 1.8.

Let $f = 2p^2x^2 + 2\quad q^2y^2 + cz^2 + 2\quad r'yz + 2sxz$, $\Omega = 1$ and $\Delta = 16$. Take $p = 3$ and $q = 11$. Then $s = 13$, $r' = 53$ and $c = 21$ so that

$$f_1 = 18x^2 + 242y^2 + 21z^2 + 106yz + 26xz.$$

The transformation

$$\begin{pmatrix} 10 & 13 & -6 \\ 3 & 4 & -2 \\ -14 & -18 & 9 \end{pmatrix}$$

reduces f_1 to

$$f_3 = 2X^2 + 2Y^2 + 5Z^2 - 2YZ - 2XZ.$$

Hence by Lemma 3.2 the class of forms represented by f_3 represents primitively no γ^2 where $\gamma \equiv 1 \pmod 4$.

Now take $p = 17$. Then $s = 127$, $r' = 109$, and $c = 77$ so that

$$f_2 = 578x^2 + 242y^2 + 77z^2 + 218yz + 254xz.$$

This reduces to

$$f_4 = X^2 + Y^2 + 16Z^2.$$

Hence by Lemma 3.2 the class of forms represented by f_4 represents primitively no γ^2 where $\gamma \equiv 3 \pmod 4$.

The characters

$$\left(\frac{-1}{109}\right) = \left(\frac{-1}{28377}\right) = 1,$$

$$\left(\frac{2}{109}\right) = \left(\frac{2}{28377}\right) = 1.$$

Hence f_1 and f_2 are in the same genus of forms and each possesses an almost characteristic of the second kind.

BIBLIOGRAPHY

Dickson, L. E. Modern Elementary Theory of Numbers. Chicago: The University of Chicago Press, 1939.

________. Studies in the Theory of Numbers. Chicago: The University of Chicago Press, 1930.

________. "Ternary Quadratic Forms and Congruences," Annals of Mathematics, Vol. 28, 1927, pp. 333-41.

Hadlock, E. H. "On the Progressions Associated with a Ternary Quadratic Form," American Journal of Mathematics, Vol. LVII, No. 2, April, 1935.

________. "Primitive Ternary Indefinite Quadratic Genera of More Than One Class." Paper read before the American Mathematical Society, Raleigh, North Carolina, November 29, 1952.

________. "Properly Primitive Ternary Indefinite Quadratic Genera of More Than One Class." Paper read before The Mathematical Association of America, Auburn, Alabama, March 14, 1953.

Jones, Burton W. "A New Definition of Genus for Ternary Quadratic Forms," Transactions of the American Mathematical Society, Vol. 33, 1931, pp. 92-110.

________. The Arithmetic Theory of Quadratic Forms. The Carus Mathematical Monographs of The American Mathematical Association, No. 10, 1950.

________. "The Regularity of a Genus of Positive Ternary Quadratic Forms, "Transactions of the American Mathematical Society, Vol. 33, 1931, pp. 111-24.

Jones, B. W., and Hadlock, E. H. "Properly Primitive Ternary Indefinite Quadratic Genera of More Than One Class," Proceedings of the American Mathematical Society, Vol. 4, No. 4, August, 1953, pp. 539-43.

Jones, B. W., and Pall, Gordon. "Regular and Semi-regular Positive Ternary Quadratic Forms," Acta Mathematica, Vol. 70, 1938, pp. 165-91.

Smith, Henry J.S. Collected Papers. Edited by J.W.L. Glaisher. Oxford, 1894.

BIOGRAPHICAL SKETCH

Paul Bryan Patterson was born in Benton County, Arkansas, November 3, 1900. He attended elementary school in Arkansas, and secondary school in Lawrence, Kansas. He did undergraduate work at the University of Kansas and at Colgate University, and received his Bachelor of Science degree from Colgate University in 1926 with a major in mathematics. He did graduate work at Columbia University and at the University of Florida, and received his Master of Arts in Education degree at the latter institution in 1948.

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Abstract of Dissertation Presented to the Graduate Council
in Partial Fulfillment of the Requirements for the
Degree of Doctor of Philosophy

ALMOST REGULAR FORMS

By
Paul Bryan Patterson

June, 1954

Almost characteristics of the first, second, and third kinds are defined. Also defined are almost regular forms and almost regular forms of the first, second, and third classes. It is shown that there exist infinitely many genera containing at least two classes of positive forms possessing an almost characteristic of the second or third kind where the invariants associated with the forms may or may not contain distinct odd prime factors to an odd power.

This dissertation was prepared under the direction of the candidate's supervisory committee and has been approved by all members of the committee. It was submitted to the Dean of the College of Arts and Sciences and to the Graduate Council and was approved as partial fulfillment of the requirements for the degree of Doctor of Philosophy.

June 7, 1954.

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